

M408M Second Midterm Exam, November 9, 2006

1. Parametrized curves. A particle follows the curve $\mathbf{r}(t) = \langle 3+2\cos(2t), 4+2\sin(2t), 3t+7 \rangle$, where t represents time. Find

- The speed of the particle at $t = \pi$.
- The unit tangent vector T at time $t = \pi$.
- The acceleration at time $t = \pi$, and
- The distance traveled (arclength, not straight-line distance) from $t = \pi$ to $t = 2\pi$.

Please put a BOX around each of your answers.

The velocity is $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -4\sin(2t), 4\cos(2t), 3 \rangle$, so the speed is $v(t) = |\mathbf{v}(t)| = 5$ and the tangent vector is $T = \mathbf{v}/v = \langle -4\sin(2t)/5, 4\cos(2t)/5, 3/5 \rangle$. Evaluated at time $t = \pi$, this gives a velocity of $\langle -4, 0, 3 \rangle$ a tangent vector of $\langle 0, 4/5, 3/5 \rangle$ and a speed of 5. The acceleration is $d\mathbf{v}/dt = \langle -8\cos(2t), -8\sin(2t), 0 \rangle = \langle -8, 0, 0 \rangle$. The distance traveled is $\int_{\pi}^{2\pi} v(t)dt = 5\pi$.

2. Partial derivatives and differentials. Consider the function $f(x, y, z) = e^x(1 + \ln(y))z^2 + xy^2$.

- Compute f_x , f_y , and f_z as functions of (x, y, z) , and evaluate them at the point $(0, 1, 3)$.

$$f_x = e^x(1 + \ln(y))z^2 + y^2 = 10, \quad f_y = e^x z^2/y + 2xy = 9, \quad \text{and} \quad f_z = 2e^x(1 + \ln(y))z = 6.$$

- Use the results of part (a) to estimate the value of $f(0.01, 0.98, 3.02)$.

$$f(0,1,3) = 9. \quad df = f_x dx + f_y dy + f_z dz = (0.01)(10) - (0.02)(9) + (0.02)6 = 0.04, \text{ so } f(0.01, 0.98, 3.02) \approx 9.04.$$

3. Suppose that $f(x, y, z)$ is a function of three variables, and that $f(1, 4, 3) = 13$, $f_x(1, 4, 3) = 2$, $f_y(1, 4, 3) = 5$, and $f_z(1, 4, 3) = -2$.

- Find the directional derivative of f at the point $(1, 4, 3)$ in the direction of the point $(3, 6, 4)$. [Note that $(3, 6, 4)$ is a point, not a vector.]

The displacement vector from $(1, 4, 3)$ to $(3, 6, 4)$ is $\langle 2, 2, 1 \rangle$, whose length is $\sqrt{4+4+1} = 3$. The direction of this vector is $\mathbf{u} = \langle 2/3, 2/3, 1/3 \rangle$. The directional derivative is $\nabla f \cdot \mathbf{u} = (2/3)2 + (2/3)5 - (1/3)2 = 4$.

- Find the equation of the tangent plane to the surface $f(x, y, z) = 13$ at the point $(1, 4, 3)$.

The normal vector is the gradient, $\langle 2, 5, -2 \rangle$, so our plane is $2x + 5y - 2z = 2(1) + 5(4) - 2(3) = 16$.

4. Max-min.

a) Find all critical points of the function $f(x, y) = x^2y^2 - 4x^2 - y^2 + 17$.

$f_x = 2xy^2 - 8x$. Setting this equal to zero gives $x = 0$ or $y^2 = 4$. Likewise, $f_y = 2x^2y - 2y$. Setting this equal to zero gives $y = 0$ or $x^2 = 1$. Since we can't have both $x = 0$ and $x^2 = 1$, and we can't have $y = 0$ and $y^2 = 4$, there are only 2 possibilities: $(0, 0)$ and $(\pm 1, \pm 2)$. That's 5 points: $(0, 0)$, $(1, 2)$, $(1, -2)$, $(-1, 2)$ and $(-1, -2)$.

b) Which of these critical points are local maxima? Minima? Saddle points?

We compute $f_{xx} = 2y^2 - 8$, $f_{yy} = 2x^2 - 2$ and $f_{xy} = 4xy$.

At $(0, 0)$, we have $f_{xx} = -8$, $f_{yy} = -2$ and $f_{xy} = 0$, so D is positive and we have a local max.

At all the other points, we have $f_{xx} = 0$, $f_{yy} = 0$ and f_{xy} nonzero, so D is negative and these are saddle points.

5. Lagrange multipliers. Find the maximum (or maxima, if there's a tie) of the function $f(x, y) = x^3 + y^3$ on the circle $x^2 + y^2 = 2$. Also, find the minimum (or minima). [Note: you do not have to classify the critical points as local maxima or minima. You just have to find the overall maximum and minimum values, and the points where these values occur.]

$f(x, y) = x^3 + y^3$, while $g(x, y) = x^2 + y^2 - 2$. $\nabla f = \langle 3x^2, 3y^2 \rangle$ while $\nabla g = \langle 2x, 2y \rangle$. Our three equations are then $3x^2 = 2\lambda x$ (so $x = 0$ or $\lambda = 3x/2$), $3y^2 = 2\lambda y$ (so $y = 0$ or $\lambda = 3y/2$) and $x^2 + y^2 = 2$. There are four possibilities:

1) $x = y = 0$. This contradicts $x^2 + y^2 = 2$, so is impossible.

2) $x = 0$ and $\lambda = 3y/2$. This gives the points $(0, \pm\sqrt{2})$, at which $f = \pm 2\sqrt{2}$.

3) $y = 0$ and $\lambda = 3x/2$, which gives the points $(\pm\sqrt{2}, 0)$ and $f = \pm 2\sqrt{2}$, and

4) $\lambda = 3x/2 = 3y/2$, in which case $x = y$. This gives the points $(1, 1)$ and $(-1, -1)$ at which $f = 2$ and -2 , respectively.

Since $2\sqrt{2} > 2$, the maximum value of $2\sqrt{2}$ is achieved at $(0, \sqrt{2})$ and $(\sqrt{2}, 0)$, while the minimum value of $-2\sqrt{2}$ is achieved at $(0, -\sqrt{2})$ and $(-\sqrt{2}, 0)$.