M346 First Midterm Exam, September 22, 2005

1. Let $V = \mathbb{R}_2[t]$ with (standard) basis $\mathcal{B} = \{1, t, t^2\}$ and let $W = M_{2,2}$ be the space of 2 by 2 real matrices with (standard) basis $\mathcal{D} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Consider the linear transformation $L(\mathbf{p}) = \begin{pmatrix} \mathbf{p}(1) - \mathbf{p}(0) & \mathbf{p}(2) - \mathbf{p}(0) \\ \mathbf{p}(-1) - \mathbf{p}(0) & \mathbf{p}(-2) - \mathbf{p}(0) \end{pmatrix}$ from V to W.

a) Find the matrix of L relative to the bases $\mathcal{B}$ and $\mathcal{D}$.

Since $L(1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $L(t) = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$ and $L(t^2) = \begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix}$, the matrix of L is

$$[L]_{\mathcal{DB}} = (\,[L(1)]_{\mathcal{D}} \quad [L(t)]_{\mathcal{D}} \quad [L(t^2)]_{\mathcal{D}}\,) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & -2 & 4 \end{pmatrix}$$

b) What is the dimension of $Ker(L)$? Find a basis for $Ker(L)$.

Row-reducing $[L]$ gives $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. There is one column without a pivot, so the dimension of $Ker(L)$ is 1. A basis is the single vector whose coordinates are $(1,0,0)^T$, namely the polynomial $p(t) = 1$.

c) What is the dimension of $Range(L)$? Find a basis for $Range(L)$.

Since there are pivots in the 2nd and 3rd columns, a basis for $Col([L])$ is given by the 2nd and 3rd columns, namely $(1,2,-1,-2)^T$ and $(1,4,1,4)^T$. This corresponds to the following basis for $Range(L)$: $\left\{ \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix} \right\}$.

2. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 7 \\ 5 & 10 & 13 & 18 \end{pmatrix}$.

a) Let $V = \{\mathbf{x} \in \mathbb{R}^4 | A\mathbf{x} = 0\}$. What is the dimension of V? Find a basis for V.

A row-reduces to $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. There are two free variables (x_2 and x_4), so V is 2-dimensional. A basis is $\{(-2,1,0,0)^T, (-1,0,-1,1)^T\}$.

b) In $\mathbb{R}^3$, consider the vectors $(1,2,5)^T$, $(2,4,10)^T$, $(3,5,13)^T$, and $(4,7,18)^T$. Are these vectors linearly independent? Do they span $\mathbb{R}^3$?

This problem is also solved by row-reducing A. Since there are columns without pivots, the vectors are not linearly independent. Since there is a row without a pivot, the vectors do not span.

c) Find a basis for the span of the four vectors of part (b).

Since the pivots are in the first and third columns, a basis is the first and third vectors, namely $(1,2,5)^T$ and $(3,5,13)^T$.

3. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 8x_1 - 10x_2 \\ 3x_1 - 3x_2 \end{pmatrix}$. On $\mathbb{R}^2$, consider the standard basis $\mathcal{E}$ and the alternate basis $\mathcal{B} = \left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix}\right\}$. Finally, let $\mathbf{v} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$.

a) Find $P_{\mathcal{E}\mathcal{B}}$, $P_{\mathcal{B}\mathcal{E}}$, $[\mathbf{v}]_{\mathcal{E}}$ and $[\mathbf{v}]_{\mathcal{B}}$.

$P_{\mathcal{D}\mathcal{B}} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, $P_{\mathcal{B}\mathcal{D}} = P_{\mathcal{D}\mathcal{B}}^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$, $[\mathbf{v}]_{\mathcal{E}} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$ and $[\mathbf{v}]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[\mathbf{v}]_{\mathcal{E}} = \begin{pmatrix} 9 \\ -2 \end{pmatrix}$.

b) Find the matrix $[L]_{\mathcal{E}}$ and the matrix $[L]_{\mathcal{B}}$.

$[L]_{\mathcal{E}} = \begin{pmatrix} L(\mathbf{e}_1) & L(\mathbf{e}_2) \end{pmatrix} = \begin{pmatrix} 8 & -10 \\ 3 & -3 \end{pmatrix}$, $[L]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[L]_{\mathcal{E}}P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$. Note that this implies that $L(\mathbf{b}_1) = 3\mathbf{b}_1$ and that $L(\mathbf{b}_2) = 2\mathbf{b}_2$, which can be easily checked.

4. The two parts of this problem are NOT connected.

a) In $\mathbb{R}_2[t]$, consider the vectors $\mathbf{b}_1 = 1 + t + 2t^2$, $\mathbf{b}_2 = 2 + 3t + 5t^2$ and $\mathbf{b}_3 = 3 + 7t + 9t^2$. Do these vectors form a basis for $\mathbb{R}_2[t]$? If so, find $[\mathbf{v}]_{\mathcal{B}}$, where $\mathbf{v} = 1 - 2t$. If not, find constants a_1, a_2, a_3, not all zero, such that $a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + a_3\mathbf{b}_3 = 0$.

Use the standard basis to convert this to a problem in $\mathbb{R}^3$, namely whether the vectors $(1,1,2)^T$, $(2,3,5)^T$ and $(3,7,9)^T$ form a basis for $\mathbb{R}^3$. Since the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 7 \\ 2 & 5 & 9 \end{pmatrix}$ row-reduces to the identity, these vectors DO form a basis, and $P_{\mathcal{E}\mathcal{B}} = A$. We compute $P_{\mathcal{B}\mathcal{E}}$ from $P_{\mathcal{E}\mathcal{B}}$ by row-reduction and get $P_{\mathcal{B}\mathcal{E}} = \begin{pmatrix} 8 & 3 & -5 \\ -5 & -3 & 4 \\ 1 & 1 & -1 \end{pmatrix}$ and compute $[\mathbf{v}]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[\mathbf{v}]_{\mathcal{E}} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$. Sure enough, you can check that $\mathbf{v} = 2\mathbf{b}_1 + \mathbf{b}_2 - \mathbf{b}_3$.

b) In $\mathbb{R}_3[t]$, let V be the set of polynomials $\mathbf{p}$ for which $\mathbf{p}(0) = \mathbf{p}(1) = 0$. Find a basis for V.

This is the kernel of the linear transformation $L : \mathbb{R}_3[t] \to \mathbb{R}^2$, $L(\mathbf{p}) = \begin{pmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{pmatrix}$, whose matrix is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$. This row-reduces to $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$, with free variables x_3 and x_4. The null space of $[L]$ has a basis $\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$, so the kernel of L has basis $\{-t + t^2, -t + t^3\}$.

5. True of False? Each question is worth 4 points. You do NOT need to justify your answers, and partial credit will NOT be given.

a) The plane $x_1 + 3x_2 - 4x_3 = 0$ is a subspace of $\mathbb{R}^3$. TRUE.

b) If A is a 3×5 matrix, then the dimension of the null space of A is at least 2.

TRUE. There are at most 3 pivots, so at least two columns don't have pivots, so there are at least two free variables.

c) Let $L : \mathbb{R}_5[t] \to \mathbb{R}^3$ be a linear transformation. If L is onto, then the kernel of L is 2-dimensional.

FALSE. Since $\mathbb{R}_5[t]$ is 6-dimensional, the kernel of L is always at least $6 - 3 = 3$ dimensional.

d) Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. If n vectors $\mathbf{d}_1, \ldots, \mathbf{d}_n$ span V, then the vectors $[\mathbf{d}_1]_{\mathcal{B}}, \ldots, [\mathbf{d}_n]_{\mathcal{B}}$ are linearly independent.

TRUE. Whenever the number of vectors is the same as the dimension of the space, either the vectors both span and are linearly independent, or fail to span and are linearly dependent. Since the vectors $\mathbf{d}_1, \ldots, \mathbf{d}_n$ are linearly independent in V, their coordinates $[\mathbf{d}_1]_{\mathcal{B}}, \ldots, [\mathbf{d}_n]_{\mathcal{B}}$ are linearly independent in $\mathbb{R}^n$.

e) Every linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^4$ is multiplication by a 5×4 matrix.

FALSE. Every linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^4$ is multiplication by a 4×5 matrix.