M346 Final Exam, December 15, 2009

1) The matrix $A = \begin{pmatrix} 1 & 3 & 2 & 5 \\ 2 & -1 & 1 & -1 \\ 0 & 1 & 2 & 3 \\ 3 & 3 & 5 & 7 \end{pmatrix}$ row-reduces to $B = \begin{pmatrix} 1 & 0 & 0 & -4/11 \\ 0 & 1 & 0 & 13/11 \\ 0 & 0 & 1 & 10/11 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

a)Find all solutions to $A\mathbf{x} = 0$.

b)Find a basis for the column space of $A$.

c)In $\mathbf{R}_3[t]$, let $V$ be the span of the vectors $\{1 + 2t + 3t^3, 3 - t + t^2 + 3t^3, 2 + t + 2t^2 + 5t^3, 5 - t + 3t^2 + 7t^3\}$. What is the dimension of $V$? Find a basis for $V$.

2. a) Find the eigenvalues of $\begin{pmatrix} 3 & -5 & 16 & 4 \\ 0 & 3 & 11 & 0 \\ 0 & 15 & -1 & 0 \\ 0 & 4 & 1 & 2 \end{pmatrix}$. You do not need to find the eigenvectors.

b) Find the eigenvalues *and eigenvectors* of $\begin{pmatrix} 3 & 8 \\ 2 & -3 \end{pmatrix}$.

3. Consider the equations

$$\begin{aligned} x_1(n+1) &= 2x_1(n) + 3x_2(n) \\ x_2(n+1) &= 2x_1(n) + x_2(n) \end{aligned}$$

a) If $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, what is $\mathbf{x}(n)$?

b) If $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, what is $\mathbf{x}(n)$?

c) Compute $A^n$, where $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$.

4. Consider the nonlinear system of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= x_1^2 + x_1x_2 - 4x_1 + x_2 + 1 \\ \frac{dx_2}{dt} &= x_2^2 + x_1 - 2x_2 \end{aligned}$$

This system of equations has a fixed point at $x_1 = x_2 = 1$.

a) Write down a linear system of equations that approximates this nonlinear system when $\mathbf{x}$ is close to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b) Diagonalize the matrix that appears in the linear equations.

c) Identify the stable, neutrally stable, and unstable modes. What is the dominant mode, and how fast does it grow or shrink? Is the system as a whole stable, neutral, or unstable near $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

5. Gram-Schmidt. In $\mathbf{R}^3$, consider the three vectors $\mathbf{x}_1 = (2,1,1)^T$, $\mathbf{x}_2 = (5,-1,3)^T$ and $\mathbf{x}_3 = (4,6,-8)^T$.

a) Use Gram-Schmidt to convert this basis to an orthogonal basis $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.

b) Decompose the vector $(1,2,3)^T$ as a linear combination of the vectors in this orthogonal basis. (Warning: the answer involves fractions.)

6. Let $V$ be the space of functions on the interval $[0,\pi]$ with boundary conditions $f(0) = 0$, $f(\pi) = 0$.

a) Let $A = 4 + \frac{d^2}{dx^2}$ be an operator on $V$. (In other words, $(Af)(x) = 4f(x) + f''(x)$) Find all the eigenvalues and eigenvectors of $A$.

b) Consider the partial differential equation

$$\frac{\partial^2 f(x,t)}{\partial t^2} = 4f(x,t) + \frac{\partial^2 f(x,t)}{\partial x^2}$$

on $[0,\pi] \times \mathbf{R}$, and with the boundary conditions $f(0,t) = f(\pi,t) = 0$ for all $t$. Find a solution to this equation with the initial conditions $f(x,0) = \sin(x) - 5\sin(3x)$, $\frac{\partial f}{\partial t}(x,0) = 3\sin(2x)$.

7. Consider the "sawtooth function", defined by $f(x) = x$ for $0 < x < 1$ and with $f(x+1) = f(x)$. (This function is discontinuous when $x$ is an integer.)

a) We write $f(x) = \sum_n \hat{f}_n \exp(2\pi i n x)$ as a Fourier series. Find the Fourier coefficients $\hat{f}_n$.

b) We can also write $f(x)$ as a sum of sines and cosines: $f(x) = \frac{a_0}{2} + \sum_n a_n \cos(2\pi n x) + \sum_n b_n \sin(2\pi n x)$. Find the coefficients $a_n$ and $b_n$.

c) Suppose that $g(x)$ is a periodic function that solves the equation $d^2g(x)/dx^2 = f(x) - \frac{1}{2}$. Find the Fourier coefficients $\hat{g}_n$ for all $n \neq 0$. ($\hat{g}_0$ is a constant of integration and is arbitrary.)

8. True or false? (2 points each, no partial credit, and no penalty for guessing.)

a) Every standing wave on the interval $[0, L]$, with Dirichlet boundary conditions, can be written as a sum of traveling waves.

b) If $R$ is a rotation in 3-dimensional space, then the trace of $R$ is at least $-1$.

c) If $B$ is a complex anti-symmetric matrix ($B^T = -B$), then $e^B$ is unitary.

d) If $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of a Hermitian matrix $A$, then $\langle \mathbf{x} | \mathbf{y} \rangle = 0$.

e) Suppose that $A$ is a $5 \times 5$ matrix with determinant 0 and trace 5. If 1 is an eigenvalue with geometric multiplicity 3 then $A$ is diagonalizable.

f) If $A$ is a $3 \times 5$ matrix and $\mathbf{b} \in \mathbf{R}^3$, then there are infinitely many solutions to $A\mathbf{x} = \mathbf{b}$.

g) If $A$ is an $m \times n$ matrix and $\mathbf{b} \in \mathbf{R}^m$, then there exists a least-squares solution to $A\mathbf{x} = \mathbf{b}$, no matter what $A$ and $\mathbf{b}$ are.

h) If $\mathcal{B}$ and $\mathcal{D}$ are different bases for a vector space $V$ and $L : V \to V$ is an operator, then $[L]_{\mathcal{B}}$ and $[L]_{\mathcal{D}}$ have the same eigenvalues.