M362K First Midterm Exam Solutions, September 29, 2010

1. The Texas Lottery is considering a new game in which players pick 7 (different) integers between 1 and 40 and the state then randomly picks 5 (different) integers between 1 and 40, say by pulling 5 balls out of a rotating bin. If your choices include all 5 of the state's numbers, you win the jackpot.

a) Suppose that the winning numbers one week are 3, 5, 18, 22, and 38. How many different player choices can win the jackpot?

You must pick the state's 5 numbers (only one way to do that) and two of the remaining 35 numbers, so the answer is $\binom{35}{2} = 595$.

b) In a different week, suppose that you pick 2, 3, 5, 8, 13, 21, and 34. How many different possibilities for the state's numbers will lead to your winning the jackpot.

The state must pick 5 of your 7 numbers, so there are $\binom{7}{5} = 21$ possibilities.

c) If you play once, what is the probability of your winning the jackpot?

This can be done two ways, either as the answer to (a) divided by the number of possible player choices, or as the answer to (b) divided by the number of possible state choices. That is, $\binom{35}{2}/\binom{40}{7}$ and $\binom{7}{5}/\binom{40}{5}$ are both correct (and are approximately 3.19×10^{-5}).

2. Two baseball teams (call them A and B) are tied for first place with 7 games left in the season. Their last 7 games are against each other, so whichever team wins the majority of games will win the division.

Suppose that team A has a 50-50 chance of winning each game, independent of all the other games.

a) What is the conditional probability that A wins the division, given that they win the first game? Simplify your answer as much as possible.

Team A must win 3 or more of the remaining 6 games, so their chance is $\left[\binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}\right]/64 = 42/64 = 21/32$, or around 65.5%.

b) What is the conditional probability that A wins the first game, given that they win the division?

If E is the event that they win the first game and F is the event that they win the series, then $P(E) = P(F) = 1/2$ (by symmetry), and we already computed $P(F|E) = 21/32$. Then $P(EF) = P(F|E)P(F) = 21/64$, and $P(E|F) = P(EF)/P(E) = 21/32$.

3. The safety record of Delaware Overseas Airways (DOA) is not the best. On any given flight, each engine has a 1/3 chance of failure, independent of any other engine. Fortunately, it only takes one good engine to fly a 2-engine plane, and it only takes two good engines to fly a 4-engine plane. (If *more* than half the engines fail, then the plane will crash and everybody on the plane will die.)

a) If you fly on a 2-engine plane, what is the probability that you will survive the flight?

You survive if 0 or 1 engines fail, which has probability $(2/3)^2+2(1/3)(2/3) = 8/9$.

b) If you fly on a 4-engine plane, what is the probability that you will survive the flight?

You survive if 0, 1 or 2 engines fail, which has probability $(2/3)^4 + 4(2/3)^3(1/3) + 6(2/3)^2(1/3)^2 = 72/81 = 8/9$. It's a coincidence that this is the same as the answer to (a). If the failure rate for an engine were less than 1/3, then the 4-engine plane would be safer, and if the failure rate were greater than 1/3, then the 2-engine plane would be safer.

In real life, planes *can* land safely with only half their engines working, and the failure rates are *much, much* lower than 1/3, so in fact larger 4-engine planes are safer than smaller 2-engine planes.

By the way, DOA more commonly stands for "dead on arrival". The name "Delaware Overseas Airways" comes from the classic newspaper parody "Not the New York Times", where an ad for DOA offered discounts on their "just short of the runway" service.

4. Suppose that I have two coins in my pocket, one regular and one 2-headed. I pull a coin out at random (50% chance for each), and then flip it twice. Let E be the event that the first toss is heads. Let F be the event that the second toss is heads. Are E and F independent? Why or why not?

These are not independent. $P(E) = 3/4$ (half of the times I pull out the ordinary coin and all the times I pull out the 2-headed coin), and likewise $P(F) = 3/4$, but $P(EF) = 5/8$ (a quarter of the times I pull out the fair coin and all the times I pull out the 2-headed coin), which is not $P(E)P(F)$.

5. A number, like 3 or 1221 or 353, that reads the same forwards and backwards, is called a pallindrome. How many pallindromes are there between 1 and 1,000?

There are 9 1-digit pallindromes, 9 2-digit pallindromes, and 90 3 digit pallindromes, since the middle digit can be anything and the outer digit can be anything but 0, so the total is 108.