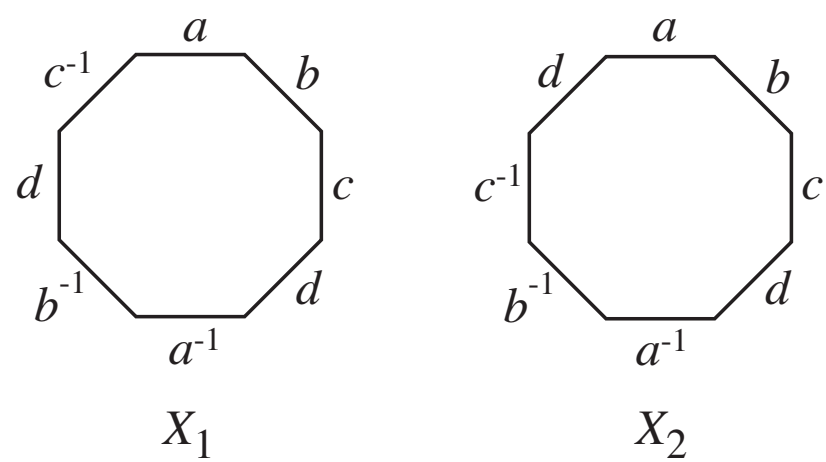
Preliminary Examination in Topology: January 2011

Algebraic Topology portion

Instructions: If possible, answer all three questions on both sides of this sheet. If this is not possible, then two complete solutions is better than three partial solutions.

Time Limit: 90 minutes.



1. Consider two surfaces X_1 and X_2, each obtained by identifying edges of an octagon as is the figure.

   (a) Is either of these surfaces a regular cover of the other? If so, indicate which one, and list all possibilities for the number of sheets.

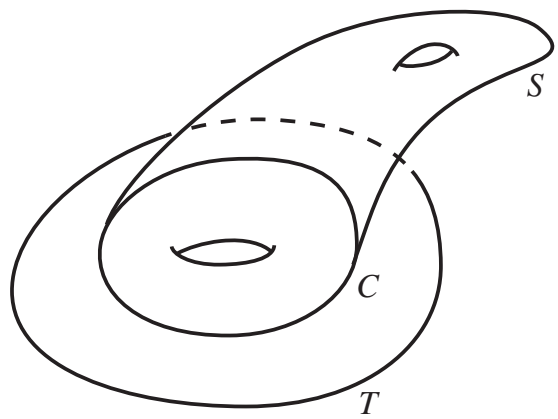
   (b) Is either of these surfaces an irregular cover of the other? If so, indicate which one, and list all possibilities for the number of sheets.

2. Give clear proofs of the following two assertions. We are looking for proofs from first principles — quoting a theorem will not suffice.

   (a) Let $f : RP^2 \to X \times Y$ be continuous and assume that $p_1 \circ f$ and $p_2 \circ f$ are each homotopic to constant maps, where $p_1 : X \times Y \to X, p_2 : X \times Y \to Y$ are the projection maps. Then f is homotopic to a constant map.

   (b) Let $p : \widetilde{X} \to X$ be a cover and $p(\tilde{x}) = x$. Let $F : D^2 \to X$ be a continuous map of a 2-dimensional disk into X with $F(y) = x$ where $y \in \partial D^2$. Then there is a lift $\widetilde{F} : D^2 \to \widetilde{X}$ of F such that $\widetilde{F}(y) = \tilde{x}$.

3. Let T be a torus and C a homotopically non-trivial, simple closed curve in T. Let X be the 2-complex obtained by attaching a 1-punctured torus, S, to T along C (that is, by identifying ∂S with C).



$X = T \cup_C S$

(a) Compute $\pi_1(X)$.
(b) Show that C must lift to a closed loop (as opposed to an open path) on any 2-fold cover of X.
(c) Give three non-homeomorphic, connected, covering spaces of X and exhibit (explain or draw) a covering map for each. You needn't show that the spaces are not homeomorphic. Are there any others?