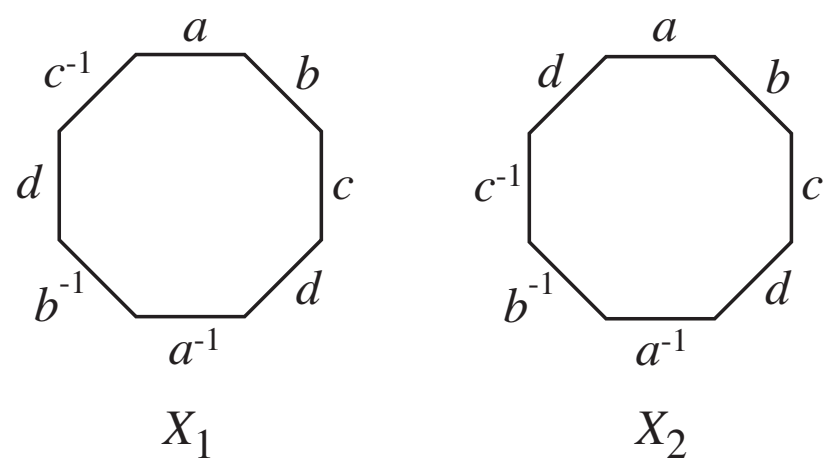
Preliminary Examination in Topology: January 2011

Algebraic Topology portion

Instructions: If possible, answer all three questions on both sides of this sheet. If this is not possible, then two complete solutions is better than three partial solutions.

Time Limit: 90 minutes.



1. Consider two surfaces X_1 and X_2, each obtained by identifying edges of an octagon as is the figure.

(a) Is either of these surfaces a regular cover of the other? If so, indicate which one, and list all possibilities for the number of sheets.

(b) Is either of these surfaces an irregular cover of the other? If so, indicate which one, and list all possibilities for the number of sheets.

Answer to both parts We identify the surfaces by orientation and Euler characteristic. X_1 has two vertices, 4 edges and one face, so $\chi(X_1) = -1$. (One vertex is the tail of a, the head of b, and the tail of c, while the other vertex is the head of a, the tail of b, the head of c, and the head and tail of d.) X_2 has only one vertex, so $\chi(X_2) = -2$. Both surfaces are non-orientable, thanks to the d edges. So $X_1 = \#_3 RP^2$ and $X_2 = \#_4 RP^2$, and X_2 is the double cover of X_1. (For instance, you can view X_1 as a big RP^2 with two small RP^2's glued in. Taking the double cover of the big RP^2 gives an S^2 with four small RP^2's glued in, which is homeomorphic to X_2.) Since all double covers are regular, X_2 is a regular double cover of X_1, with two sheets, but neither surface is an irregular cover of the other.

2. Give clear proofs of the following two assertions. We are looking for proofs from first principles — quoting a theorem will not suffice.

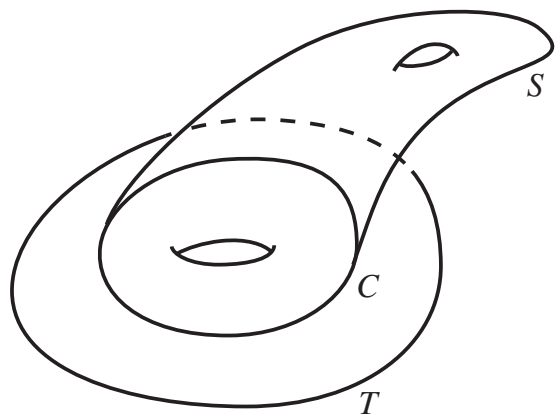
(a) Let $f : RP^2 \to X \times Y$ be continuous and assume that $p_1 \circ f$ and $p_2 \circ f$ are each homotopic to constant maps, where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the projection maps. Then f is homotopic to a constant map.

Answer Let $f_i = p_i \circ f$, and let $F_1 : RP^2 \times [0,1] \to X$,$F_2 : RP^2 \times [0,1] \to Y$ be homotopies of f_1 and f_2 to the a constant map. Then $F = F_1 \times F_2$ is a null-homotopy of f. To see that F is continuous, note that $(F_1 \times F_2)^{-1}(U \times V) = F_1^{-1}(U) \cap F_2^{-1}(V)$.

(b) Let $p : \widetilde{X} \to X$ be a cover and $p(\tilde{x}) = x$. Let $F : D^2 \to X$ be a continuous map of a 2-dimensional disk into X with $F(y) = x$ where $y \in \partial D^2$. Then there is a lift $\widetilde{F} : D^2 \to \widetilde{X}$ of F such that $\widetilde{F}(y) = \tilde{x}$.

Answer Represent D^2 by the square $[0,1] \times [0,1]$ with $y = (0,0)$. Let $\epsilon > 0$ be such that if $A \subset [0,1] \times [0,1]$ has diameter at most ϵ then $F(A)$ lies in an open set of X that is evenly covered (ϵ is the Lebesgue number of the cover of $[0,1] \times [0,1]$ given by $\{F^{-1}(U) | U$ is an evenly covered open subset of X$\}$.) Subdivide $[0,1] \times [0,1]$ into sub-rectangles of dimensions $\epsilon/2 \times \epsilon/2$. We construct the lift $\widetilde{F}$ one subrectangle at a time. Start with $[0, \epsilon/2] \times [0, \epsilon/2] = A$. Let U_A be an open set of X evenly covered by p and s.t. $F(A) \subset U_A$. Let $\widetilde{U_A}$ be the slice of this even cover containing $\tilde{x}$. Lift $\widetilde{F}|_A$ as $\tilde{p}|_{\widetilde{U_A}}^{-1} \circ F|_A$. Note that $\widetilde{F}|_A(0,0) = \tilde{x}$. We follow the same procedure for each subrectangle. Each time we pick a new subrectangle that intersects the previously defined domain along either its left-hand edge, its bottom edge, or the union of the two edges. As this overlap is connected, the overlap will map into a single slice above an evenly covered open set containing the image of f restricted to this subrectangle. We extend the map over this rectangle in this slice as above. We continue until $\widetilde{F}$ has been defined over D^2.

3. Let T be a torus and C a homotopically non-trivial, simple closed curve in T. Let X be the 2-complex obtained by attaching a 1-punctured torus, S, to T along C (that is, by identifying ∂S with C).



$X = T \cup_C S$

(a) Compute $\pi_1(X)$.

Answer Use Van Kampen's theorem, with U being a neighborhood of T and V being a neighborhood of S. Since $\pi_1(U) =< a, b|ab = ba >$, where a is the class of C, since $\pi_1(V) =< e, f >$, and since the class of C in $\pi_1(V)$ is $efe^{-1}f^{-1}$, $\pi_1(X) =< a, b, e, f|ab = ba, a = efe^{-1}f^{-1} >$.

(b) Show that C must lift to a closed loop (as opposed to an open path) on any 2-fold cover of X.

Answer Any 2-fold cover, $p : \tilde{X} \to X$ is regular, hence its fundamental group is the kernel of a map $f : \pi_1(X) \to Z/2Z$. By part (a), the class of C in $\pi_1(X)$ is a commutator and hence in the kernel of f. Since the class of C in $\pi_1(X)$ in in $p_*(\pi_1(\tilde{X})$, C lifts to a closed loop.

(c) Give three non-homeomorphic, connected, covering spaces of X and exhibit (explain or draw) a covering map for each. You needn't show that the spaces are not homeomorphic. Are there any others?

Answer Each double cover of X will be gotten by gluing a (possibly disconnected) double cover of T to a (possibly disconnected) double cover of S along two copies of C to obtain a connected space.

There are two double covers of T in which the preimage of C is two curves:

(1) A single torus to which C lifts to two parallel curves.
(2) Two homeomorphic copies of T.

Likewise, there are two possible double covers of S:

(3) A twice-punctured torus to which C lifts to the two punctures.
(4) Two homeomorphic copies of S.

1 and 3, 1 and 4, and 2 and 3 give non-homeomorphic connected covers. 2 and 4 gives a disconnected space, which doesn't count. These are all of the double covers.