1. Consider two surfaces $X_1$ and $X_2$, each obtained by identifying edges of an octagon as is the figure.

   (a) Is either of these surfaces a regular cover of the other? If so, indicate which one, and list all possibilities for the number of sheets.
   
   (b) Is either of these surfaces an irregular cover of the other? If so, indicate which one, and list all possibilities for the number of sheets.

**Answer to both parts** We identify the surfaces by orientation and Euler characteristic. $X_1$ has two vertices, 4 edges and one face, so $\chi(X_1) = -1$. (One vertex is the tail of $a$, the head of $b$, and the tail of $c$, while the other vertex is the head of $a$, the tail of $b$, the head of $c$, and the head and tail of $d$.) $X_2$ has only one vertex, so $\chi(X_2) = -2$. Both surfaces are non-orientable, thanks to the $d$ edges. So $X_1 = \#_3\mathbb{RP}^2$ and $X_2 = \#_4\mathbb{RP}^2$, and $X_2$ is the double cover of $X_1$. (For instance, you can view $X_1$ as a big $\mathbb{RP}^2$ with two small $\mathbb{RP}^2$’s glued in. Taking the double cover of the big $\mathbb{RP}^2$ gives an $S^2$ with four small $\mathbb{RP}^2$’s glued in, which is homeomorphic to $X_2$.) Since all double covers are regular, $X_2$ is a regular double cover of $X_1$, with two sheets, but neither surface is an irregular cover of the other.
2. Give clear proofs of the following two assertions. We are looking for proofs from first principles — quoting a theorem will not suffice.

(a) Let \( f : RP^2 \to X \times Y \) be continuous and assume that \( p_1 \circ f \) and \( p_2 \circ f \) are each homotopic to constant maps, where \( p_1 : X \times Y \to X \) and \( p_2 : X \times Y \to Y \) are the projection maps. Then \( f \) is homotopic to a constant map. 

**Answer** Let \( f_i = p_i \circ f \), and let \( F_1 : RP^2 \times [0, 1] \to X, F_2 : RP^2 \times [0, 1] \to Y \) be homotopies of \( f_1 \) and \( f_2 \) to the a constant map. Then \( F = F_1 \times F_2 \) is a null-homotopy of \( f \). To see that \( F \) is continuous, note that \((F_1 \times F_2)^{-1}(U \times V) = F_1^{-1}(U) \cap F_2^{-1}(V)\).

(b) Let \( p : \tilde{X} \to X \) be a cover and \( p(\tilde{x}) = x \). Let \( F : D^2 \to X \) be a continuous map of a 2-dimensional disk into \( X \) with \( F(y) = x \) where \( y \in \partial D^2 \). Then there is a lift \( \tilde{F} : D^2 \to \tilde{X} \) of \( F \) such that \( \tilde{F}(y) = \tilde{x} \).

**Answer** Represent \( D^2 \) by the square \([0, 1] \times [0, 1] \) with \( y = (0, 0) \). Let \( \epsilon > 0 \) be such that if \( A \subset [0, 1] \times [0, 1] \) has diameter at most \( \epsilon \) then \( F(A) \) lies in an open set of \( X \) that is evenly covered (\( \epsilon \) is the Lebesgue number of the cover of \([0, 1] \times [0, 1] \) given by \( \{F^{-1}(U)\mid U \text{ is an evenly covered open subset of } X\} \)). Subdivide \([0, 1] \times [0, 1] \) into sub-rectangles of dimensions \( \epsilon/2 \times \epsilon/2 \). We construct the lift \( \tilde{F} \) one subrectangle at a time. Start with \([0, \epsilon/2] \times [0, \epsilon/2] = A \). Let \( U_A \) be an open set of \( X \) evenly covered by \( p \) and s.t. \( F(A) \subset U_A \). Let \( \tilde{U}_A \) be the slice of this even cover containing \( \tilde{x} \). Lift \( \tilde{F}|_A \) as \( \tilde{p}|_{\tilde{U}_A}^{-1} \circ F|_A \). Note that \( \tilde{F}|_A(0, 0) = \tilde{x} \). We follow the same procedure for each subrectangle. Each time we pick a new subrectangle that intersects the previously defined domain along either its left-hand edge, its bottom edge, or the union of the two edges. As this overlap is connected, the overlap will map into a single slice above an evenly covered open set containing the image of \( f \) restricted to this subrectangle. We extend the map over this rectangle in this slice as above. We continue until \( \tilde{F} \) has been defined over \( D^2 \)
3. Let $T$ be a torus and $C$ a homotopically non-trivial, simple closed curve in $T$. Let $X$ be the 2-complex obtained by attaching a 1-punctured torus, $S$, to $T$ along $C$ (that is, by identifying $\partial S$ with $C$).

![Diagram](image_url)

\[ X = T \cup_C S \]

(a) Compute $\pi_1(X)$.

**Answer** Use Van Kampen’s theorem, with $U$ being a neighborhood of $T$ and $V$ being a neighborhood of $S$. Since $\pi_1(U) = \langle a, b| ab = ba \rangle$, where $a$ is the class of $C$, since $\pi_1(V) = \langle e, f \rangle$, and since the class of $C$ in $\pi_1(V)$ is $efe^{-1}f^{-1}$, $\pi_1(X) = \langle a, b, e, f| ab = ba, a = efe^{-1}f^{-1} \rangle$.

(b) Show that $C$ must lift to a closed loop (as opposed to an open path) on any 2-fold cover of $X$.

**Answer** Any 2-fold cover, $p : \tilde{X} \to X$ is regular, hence its fundamental group is the kernel of a map $f : \pi_1(X) \to Z/2Z$. By part (a), the class of $C$ in $\pi_1(X)$ is a commutator and hence in the kernel of $f$. Since the class of $C$ in $\pi_1(X)$ in in $p_*(\pi_1(\tilde{X}))$, $C$ lifts to a closed loop.

(c) Give three non-homeomorphic, connected, covering spaces of $X$ and exhibit (explain or draw) a covering map for each. You needn’t show that the spaces are not homeomorphic. Are there any others?

**Answer** Each double cover of $X$ will be gotten by gluing a (possibly disconnected) double cover of $T$ to a (possibly disconnected) double cover of $S$ along two copies of $C$ to obtain a connected space.

There are two double covers of $T$ in which the preimage of $C$ is two curves:

1. A single torus to which $C$ lifts to two parallel curves.
2. Two homeomorphic copies of $T$.

Likewise, there are two possible double covers of $S$:

3. A twice-punctured torus to which $C$ lifts to the two punctures.
4. Two homeomorphic copies of $S$.

1 and 3, 1 and 4, and 2 and 3 give non-homeomorphic connected covers. 2 and 4 gives a disconnected space, which doesn’t count. These are all of the double covers.