1. (30 points) (a) Let $X$ be a punctured torus and let $\gamma$ be a loop around the puncture, as depicted on the blackboard. Let $x_0$ be a point on $\gamma$. Compute $\pi_1(X, x_0)$ and express the class of $\gamma$ in terms of the generators and relations of that group.

This was essentially a homework problem (not to mention part of last year’s exam). $X$ retracts to the wedge of two circles, so $\pi_1(X, x_0) = F_2 = \langle a, b \rangle$ and $[\gamma]$ is the boundary word $aba^{-1}b^{-1}$.

(b) Now let $Y$ be the union of $X$ with a disk, as shown on the board. Compute $\pi_1(Y, x_0)$ and compute the class of $\gamma$ in $\pi_1(Y, x_0)$.

Let $D$ be the disk and let $V$ be a neighborhood of the disk, and let $U$ be a neighborhood of the complement of $D$. $V$ deformation retracts to $D$, and hence to a points, while $U$ deformation retracts to $X$. By van Kampen, since $V$ is contractible, $\pi_1(Y)$ is the quotient of $\pi_1(U)$ by the image of $\pi_1(U \cap V)$ in $\pi_1(U)$. Since $U \cap V$ retracts to a circle going around the torus once, (call that $b$), $\pi_1(Y, x_0) = \langle a, b | b \rangle = \langle a \rangle = \mathbb{Z}$. The class of $\gamma$ is now the image of $aba^{-1}b^{-1}$ when we set $b = 1$, namely $[\gamma] = 1$.

(c) Now let $Z$ be the union of $T^2 \# T^2$ and two disks, as shown on the board. Compute $\pi_1(Z)$.

Again we use van Kampen, now with $U$ and $V$ being the left and right halves of the picture (overlapping slightly, of course). $U$ and $V$ are both homeomorphic to $Y$, so $\pi_1(U) = \pi_1(V) = \mathbb{Z}$, while $U \cap V$ is an annulus, with $\pi_1(U \cap V) = \mathbb{Z}$. The generator of $\pi_1(U \cap V)$ is the class of $\gamma$, which maps to the identity in both $\pi_1(U)$ and $\pi_1(V)$, so van Kampen says that $\pi_1(Z) = \mathbb{Z} \ast \mathbb{Z} = F_2$. Note that the image of $\gamma$ really matters in this calculation!
2. (20 points) The infinite dihedral group $D_{\infty}$ is a group of symmetries of the real line generated by a translation $t(x) = x + 1$ and a reflection $r(x) = -x$. Show that $D_{\infty}$ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_2$.

Let $a, b$ be the generators of $\mathbb{Z}_2 * \mathbb{Z}_2$, with $a^2 = b^2 = 1$. We map $\mathbb{Z}_2 * \mathbb{Z}_2$ to $D_{\infty}$ by $a \rightarrow r$ (that is $a(x) = -x$) and $b \rightarrow tr$ (that is, $b(x) = 1 - x$). This is well-defined since $a$ and $b$ both go to elements of order 2, so both define maps $\mathbb{Z}_2 \rightarrow D_{\infty}$, and, by the universal property of free products, any two maps from $\mathbb{Z}_2 \rightarrow D_{\infty}$ induce a (unique) map $\mathbb{Z}_2 * \mathbb{Z}_2 \rightarrow D_{\infty}$. I claim this map is an isomorphism.

It’s certainly onto, since the image of $ba$ is $t$ and the image of $a$ is $r$. We must show that it’s 1–1. Note that the image of $ab$ is $t^{-1}$. We showed in homework that all elements of $\mathbb{Z}_2 * \mathbb{Z}_2$ are of the form $(ab)^n$, $(ab)^na$, $(ba)^n$ or $(ba)^nb$ with $n \geq 0$, as these are all the reduced words. These four classes map to $t^{-n}$, $t^{-n}r = rt^n$, $t^n$ and $t^{n+1}r = rt^{-(n+1)}$, which are all distinct (except for $t^0$ and $t^{-0}$, which come from $(ab)^0$ and $(ba)^0$, which are of course the same). QED.