6 Pillars of Calculus

1. "Close is good enough"

If you can't get an exact answer right away, get an approximate answer, then a better answer, etc, and take a limit.

2. "Track the changes."

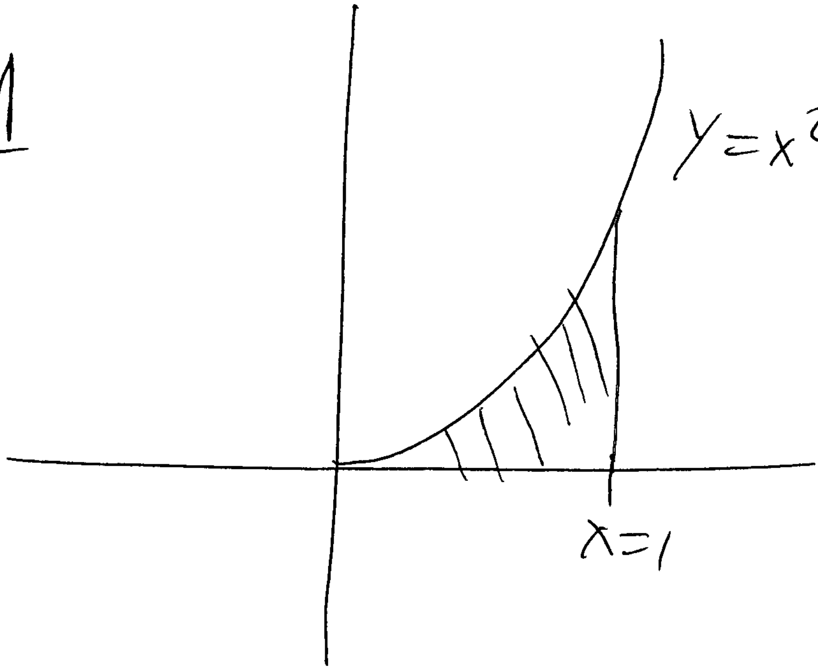
3. "What goes up, stops before coming down."

4. The whole is the sum of the parts.

5. The whole change is the sum of the partial changes.

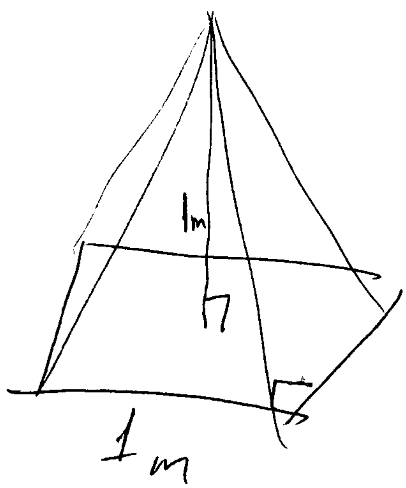
6. One variable at a time

1

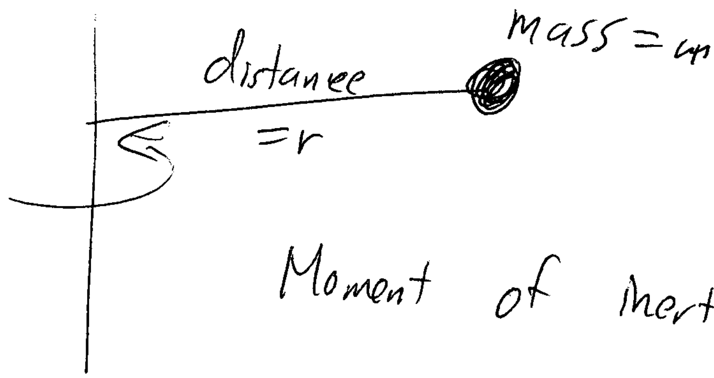


Find the area ~~of~~ under the curve $y = x^2$ between $x = 0$ and $x = 1$.

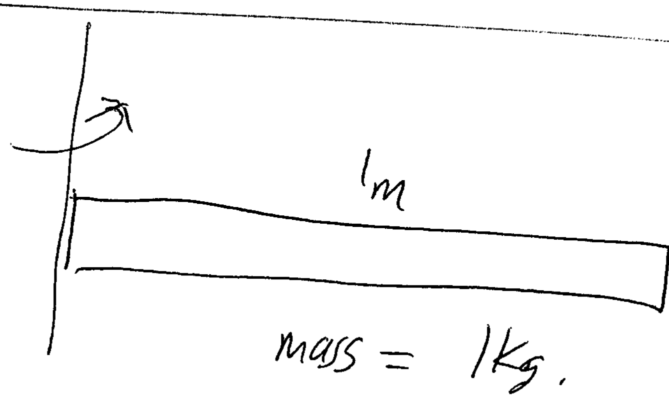
2



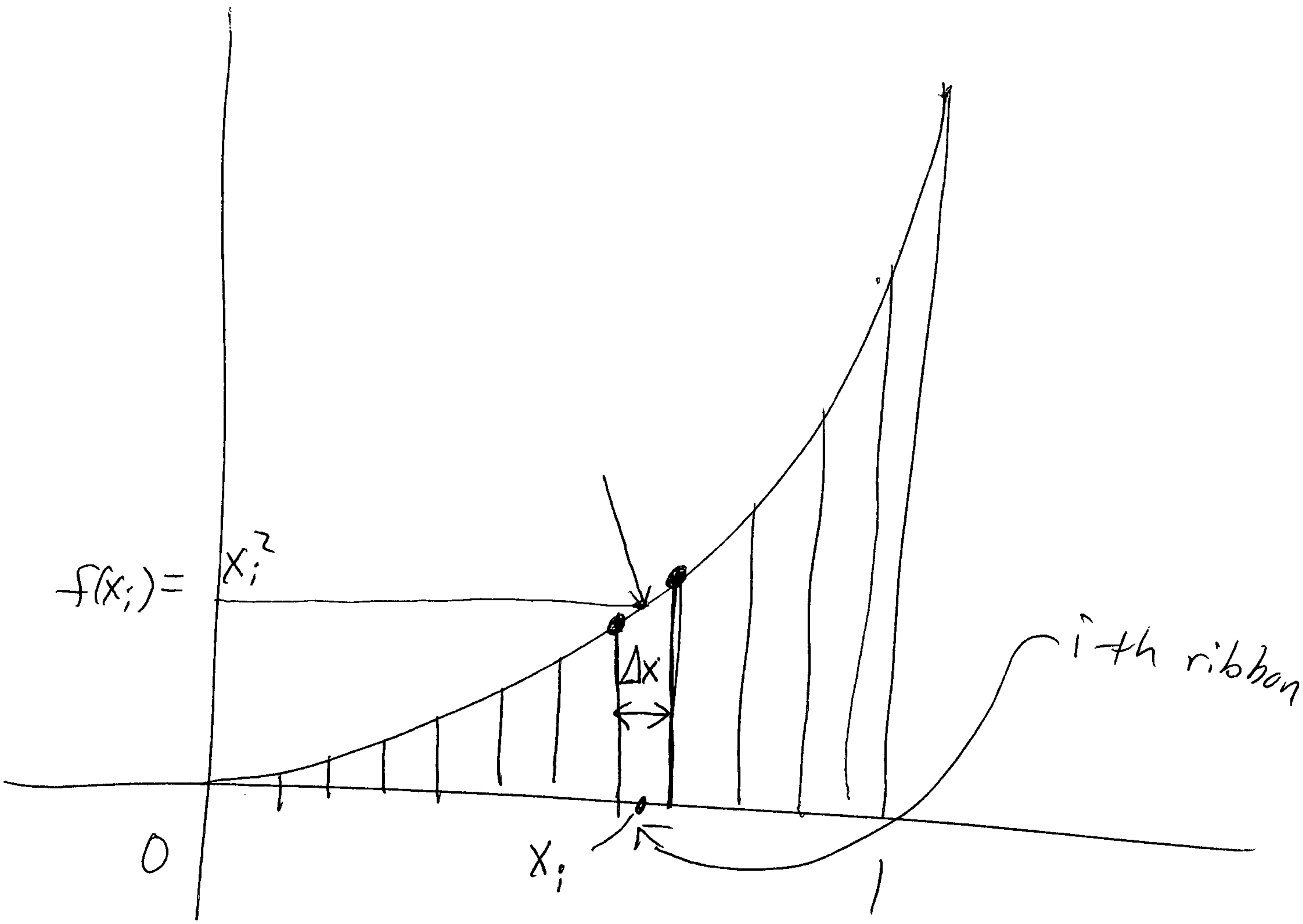
What is the volume of a pyramid with height 1m and base a 1m x 1m square?



Moment of inertia = mr^2.



What's the moment of inertia of the bar?

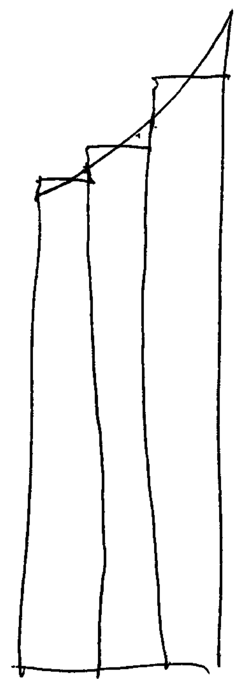


Whole area = sum of area of each slice.

Area of slice $\approx$ height $\times$ width
$= f(x_i)\,\Delta x$

The thinner the slices, the more accurate the method.

$$\text{Total area} \approx \sum_{i=1}^{N} f(x_i)\,\Delta x$$

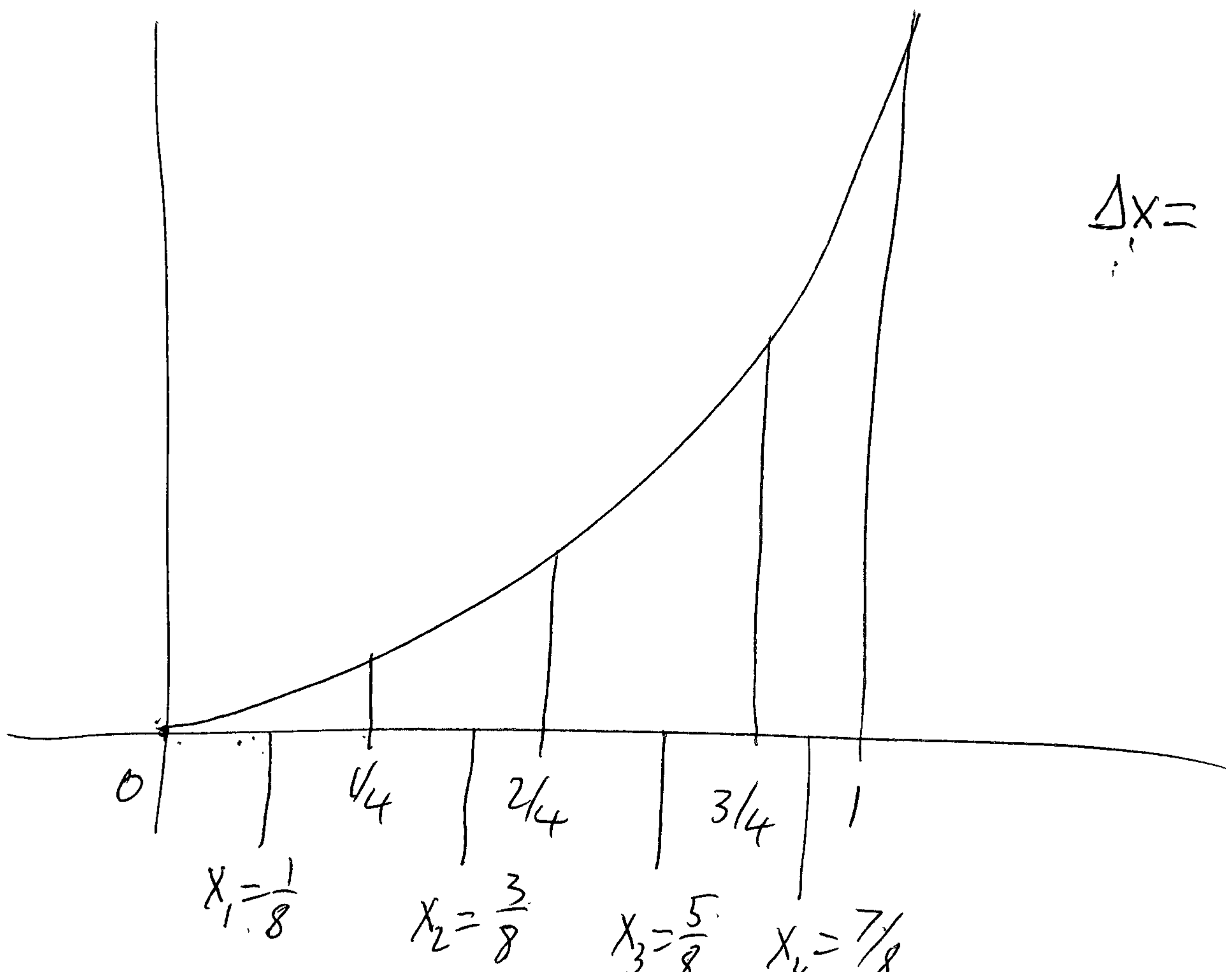


Exact area $= \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i) \Delta x$

$$\sum_{i=1}^{4} (i+1)^2 = \text{Sum as } i \text{ goes from } 1 \text{ to } 4 \text{ of } (i+1)^2$$

$$= (1+1)^2 + (2+1)^2 + (3+1)^2 + (4+1)^2$$

$$= 4 + 9 + 16 + 25 = 54$$



$$\Delta x = \frac{1}{4}$$

$$x_i = \frac{(2i-1)}{8} \qquad f(x_i) = \frac{(2i-1)^2}{8^2} \cong \text{height of } i\text{-th slice}$$

$$\text{Area of } i\text{-th slice} \cong \frac{(2i-1)^2}{8^2} \cdot \frac{1}{4}$$

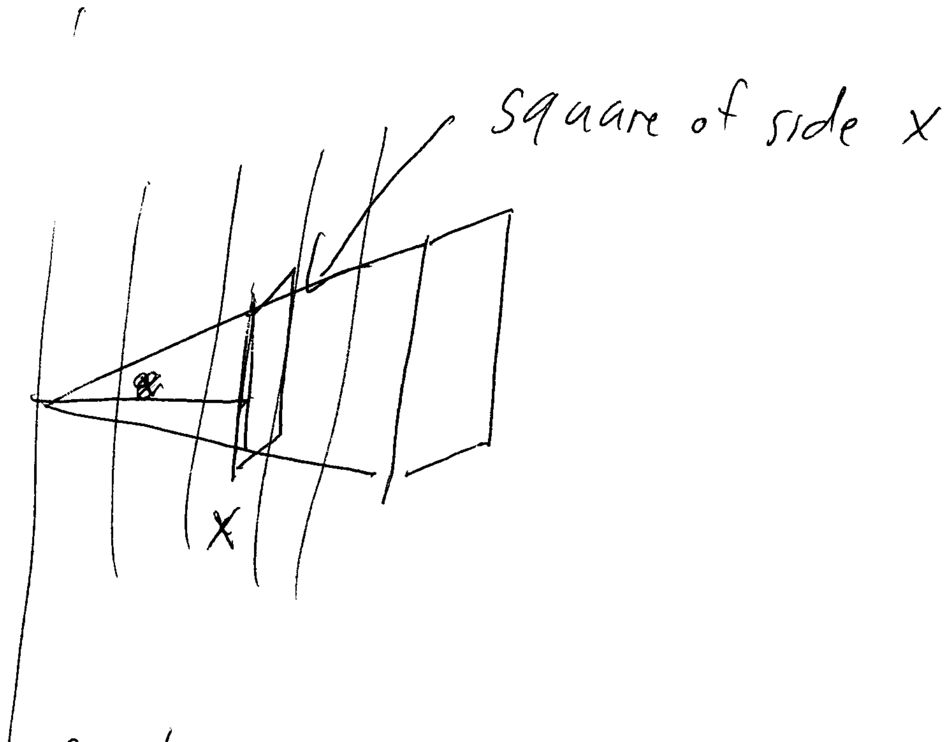
total

$$\text{Area} \approx \sum_{i=1}^{4} (x_i)^2 \Delta x = \sum_{i=1}^{4} \left(\frac{2i-1}{8}\right)^2 \cdot \frac{1}{4}$$

$$= \frac{1}{256} \sum_{i=1}^{4} (2i-1)^2$$

$$= \frac{1}{256} (1+9+25+49)$$

$$= \frac{1}{256} \cdot 84 = \frac{21}{64}$$

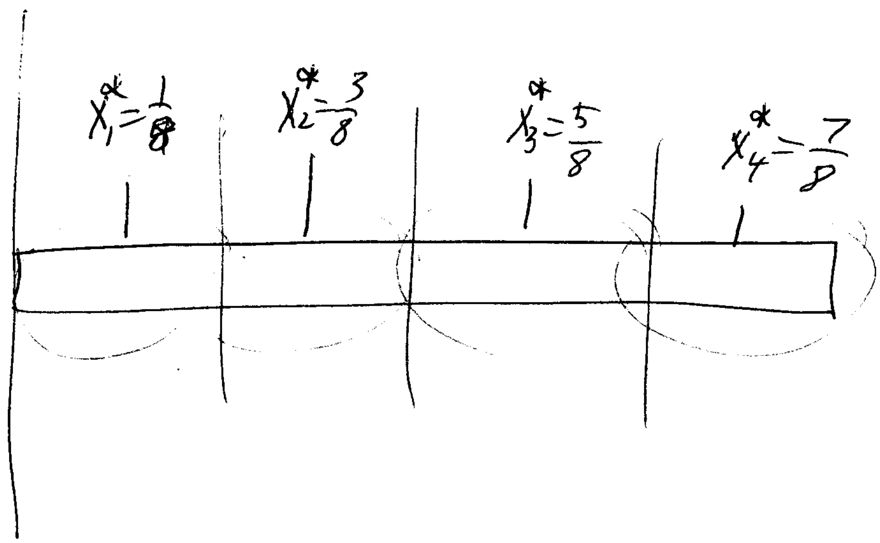


Area of slice at position $x_i \approx x_i^2$

Thickness $= \Delta x$

Volume $= x_i^2 \Delta x$

Total volume $\approx \sum_{i=1}^{N} x_i^2 \Delta x$



Total moment of inertia

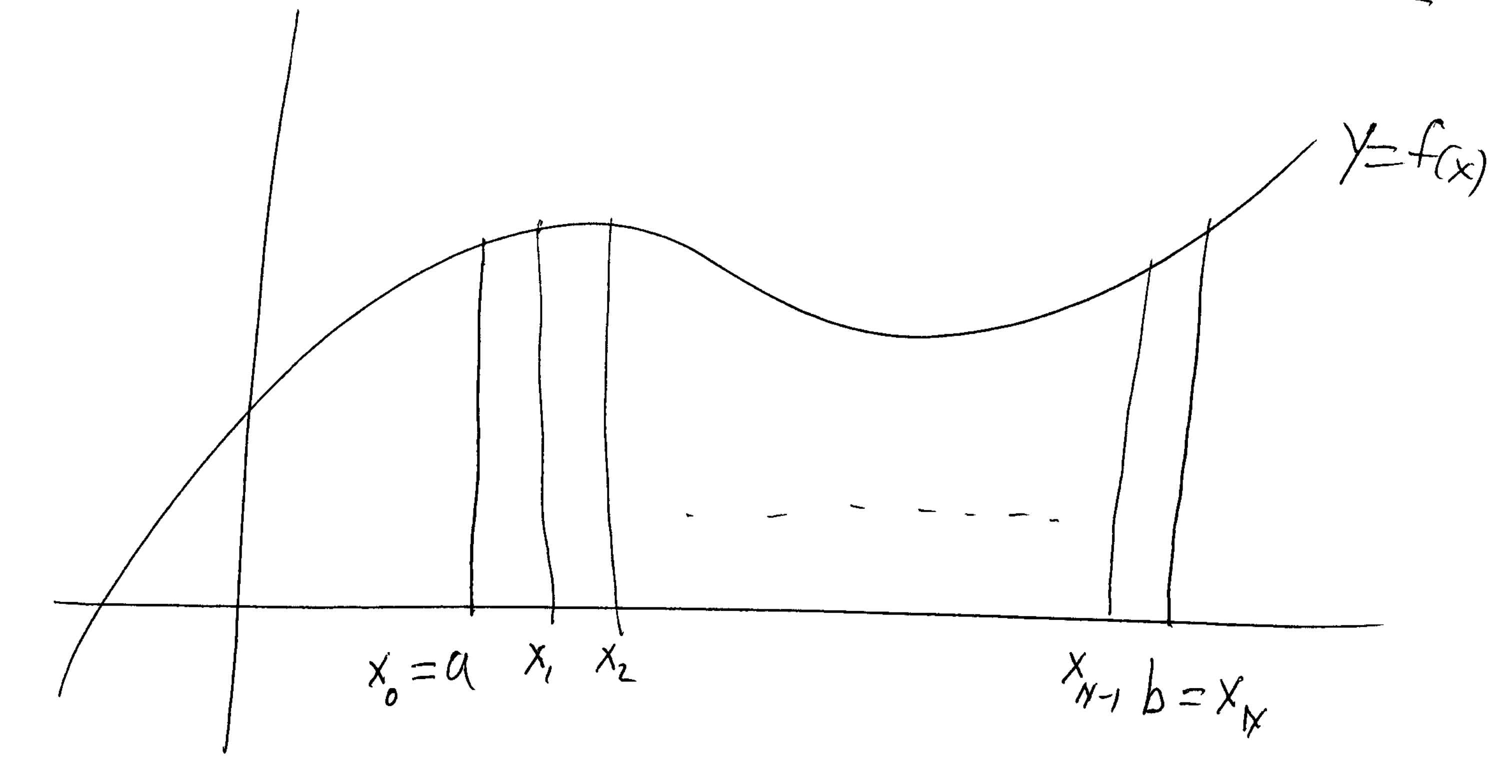
$= \sum$ moment of inertia of each piece

$$\approx \frac{1}{4}\left(\frac{1}{8}\right)^2 + \frac{1}{4}\left(\frac{3}{8}\right)^2 + \frac{1}{4}\left(\frac{5}{8}\right)^2 + \frac{1}{4}\left(\frac{7}{8}\right)^2$$

$$= \frac{21}{64}$$

Goal: Define $\int_a^b f(x)\,dx$ (definite)

= integral of $f(x)$ from a to b.



Step 1: Divide the interval $[a,b]$ into N pieces.

$$\Delta x = \frac{b-a}{N}$$

$$x_0=a,\ x_1 = a+\Delta x,\ x_2 = a+2\Delta x, \ldots$$

$$x_i = a + i(\Delta x) = a + i\left(\frac{b-a}{N}\right)$$

Step 2: Pick a sample point x_i^* in the i-th interval.

Common choices:

a) R endpoint: $x_i^* = x_i = a + i\frac{(b-a)}{N}$

b) L endpoint: $x_i^* = x_{i-1} = a + (i-1)\frac{(b-a)}{N}$

c) midpoint $x_i^* = \frac{x_i + x_{i-1}}{2}$

$= a + \frac{(2i-1)(b-a)}{2N}$.

d) maximum

e) minimum

Step 3: Contribution of each ~~tota~~ interval

$$f(x_i^*)\Delta x$$

<u>Step 4:</u> Add them up.

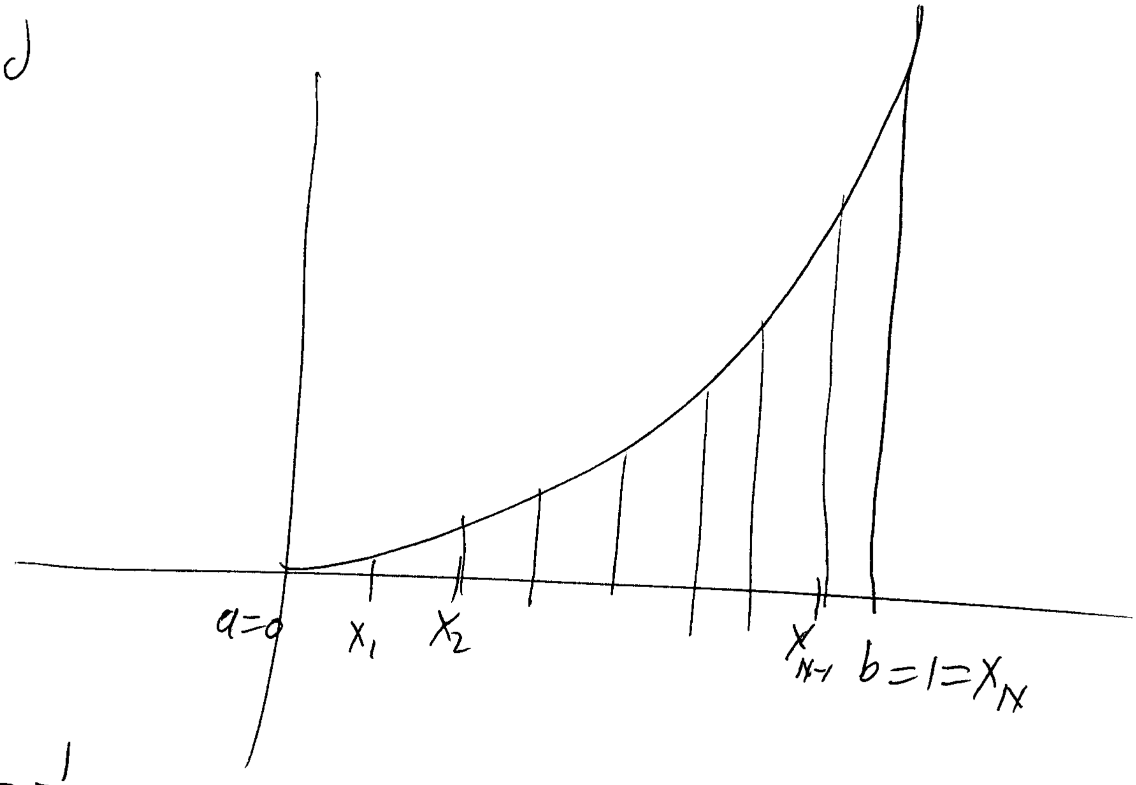
$$\text{Total} \triangleq \sum_{i=1}^{N} f(x_i^*)\,\Delta x$$

Riemann sum

Step 5: Take a limit

$$\int_a^b f(x)\,dx \overset{\text{def}}{=} \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i^*)\Delta x$$

$$\int_0^1 x^2\,dx$$



$$\Delta x = \frac{1}{N}$$

$$x_i = i\,\Delta x = \frac{i}{N}$$

Use R endpoints, $x_i^* = x_i = \frac{i}{N}$

$$\sum_{i=1}^{N} \underbrace{\left(\frac{i}{N}\right)^2}_{f(x_i^*)} \cdot \underbrace{\frac{1}{N}}_{\Delta x} = \frac{1}{N^3} \sum_{i=1}^{N} i^2$$

$$= \frac{1}{N^3}\left(1^2 + 2^2 + 3^2 + \cdots + N^2\right)$$

$$= \frac{1}{N^3}\left(\frac{N(N+1)(2N+1)}{6}\right)$$

$$\int_0^1 x^2\,dx = \lim_{N\to\infty} \frac{N(N+1)(2N+1)}{6N^3}$$

$$= \lim_{N\to\infty} \frac{2N^3 + \text{stuff}}{6N^3} = \frac{1}{3}$$