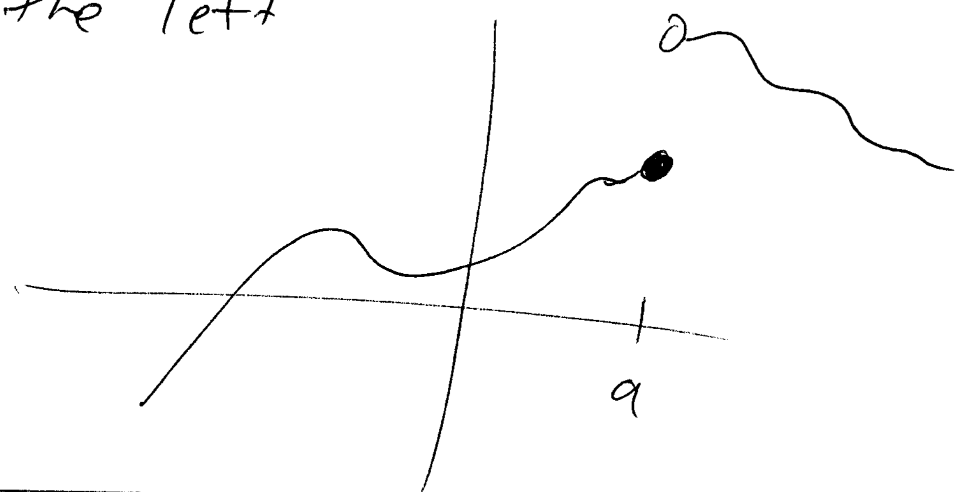
How does a function behave around $x=a$?

$$\lim_{x \to a^-} f(x) \stackrel{?}{=} f(a) \stackrel{?}{=} \lim_{x \to a^+} f(x)$$

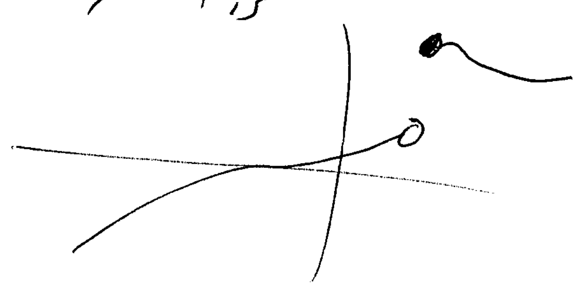
Defi: f is continuous at a if and only if (iff)

$$\lim_{x \to a} f(x) = f(a)$$

If $\lim_{x \to a^-} f(x) = f(a)$ we say f is continuous on the left

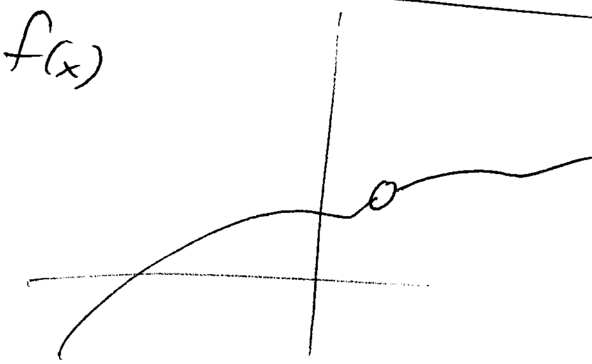


If $\lim_{x \to a^+} f(x) = f(a)$ we say f is continuous on the right.



A jump discontinuity is where
$\lim_{x \to a^{\pm}} f(x)$ exist but are different.

If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

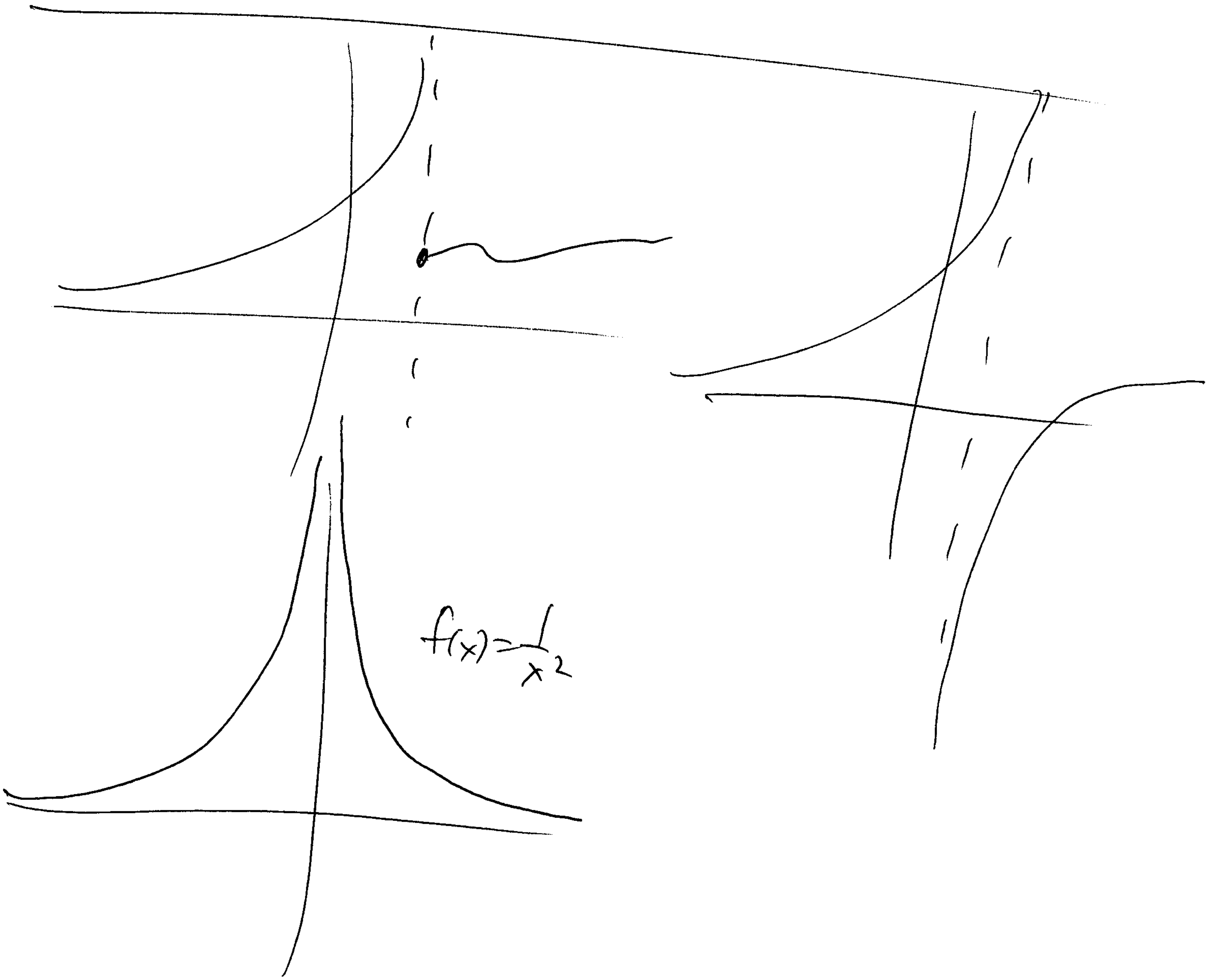


$\lim_{x \to a} f(x)$ exists.

Removable discontinuity.

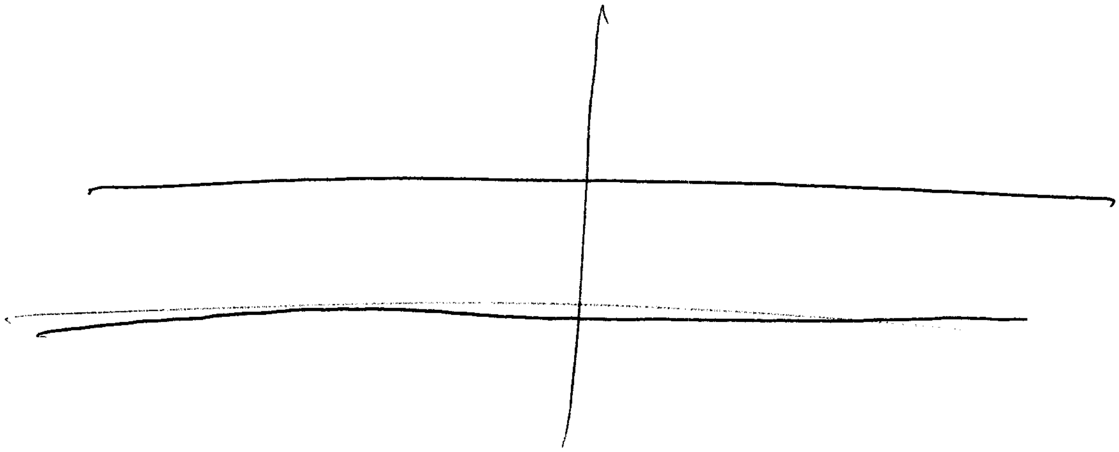
If $\lim\limits_{x \to a^-} f(x) = \infty$ or $-\infty$

we say f has an infinite discontinuity at a.



$f(x) = \frac{1}{x^2}$

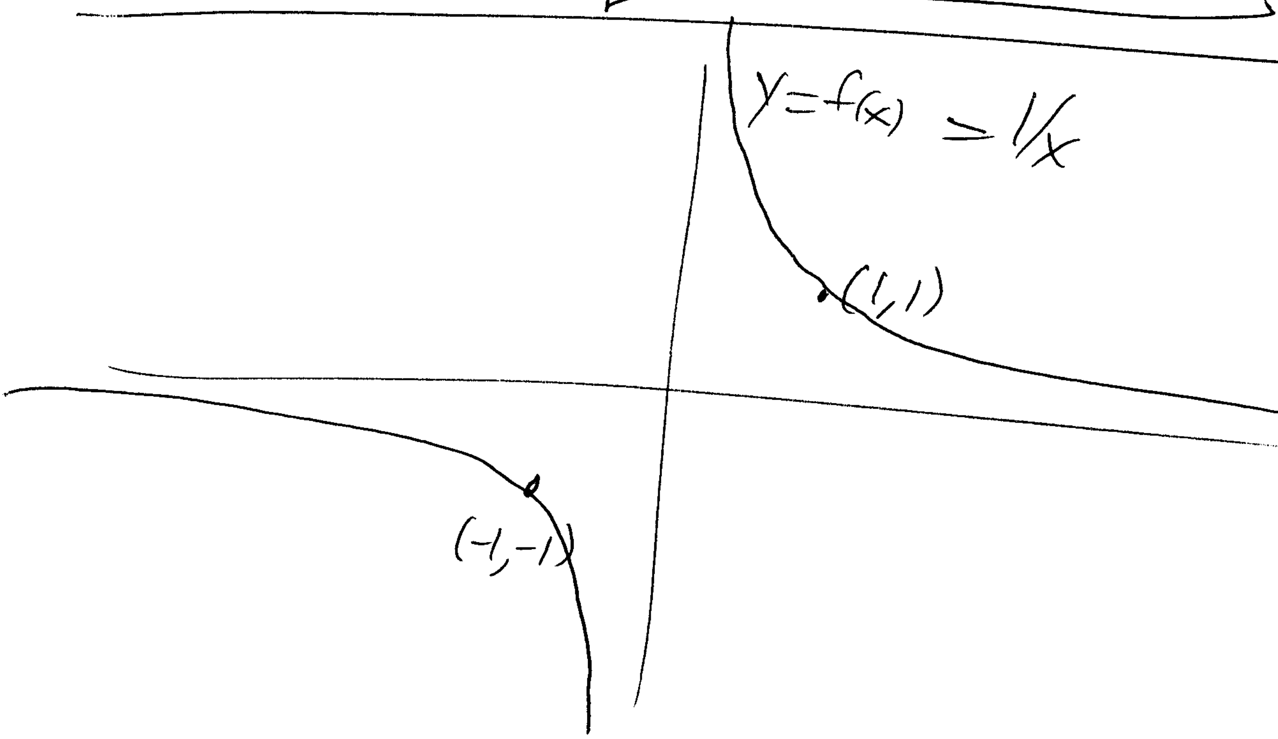
Let $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$



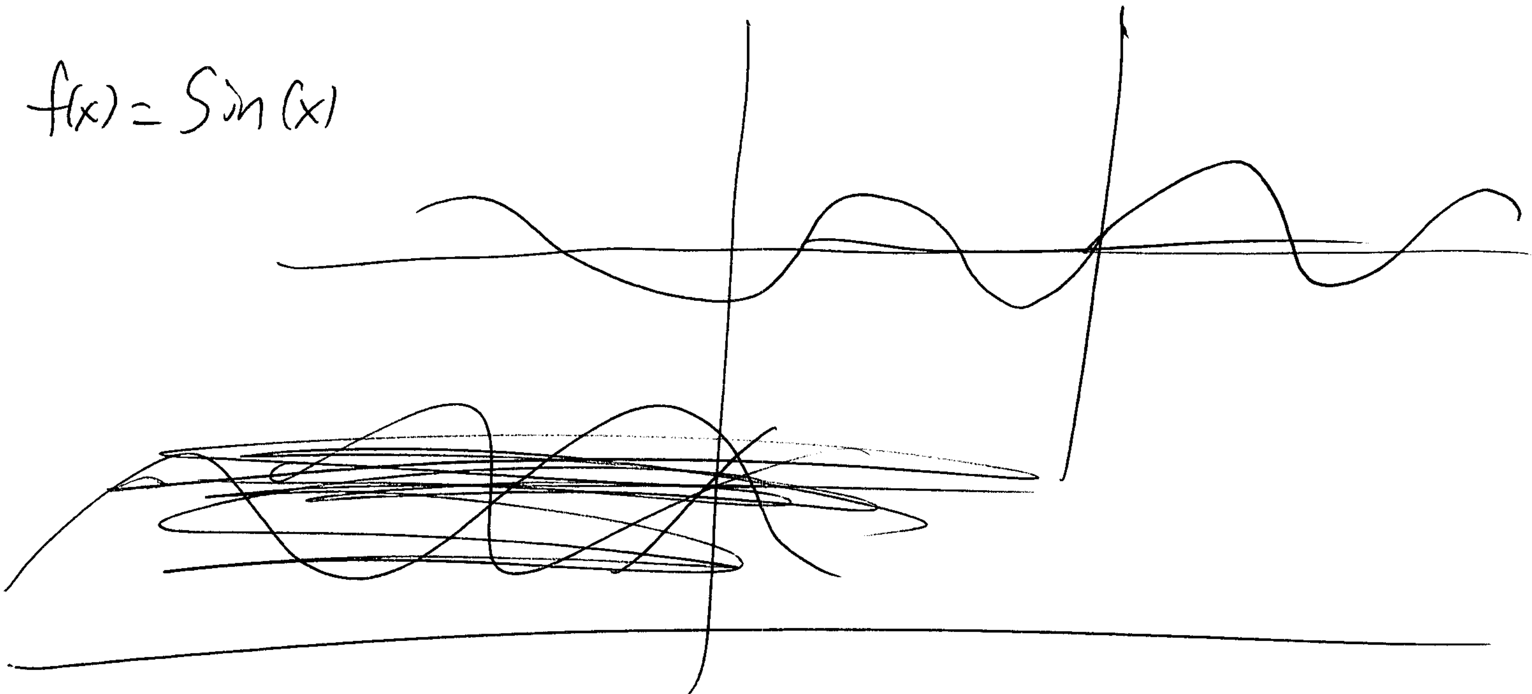
X is continuous

If f, g continuous, $f+g$, $f \cdot g$ cont.

f/g continuous as long as $g \neq 0$.

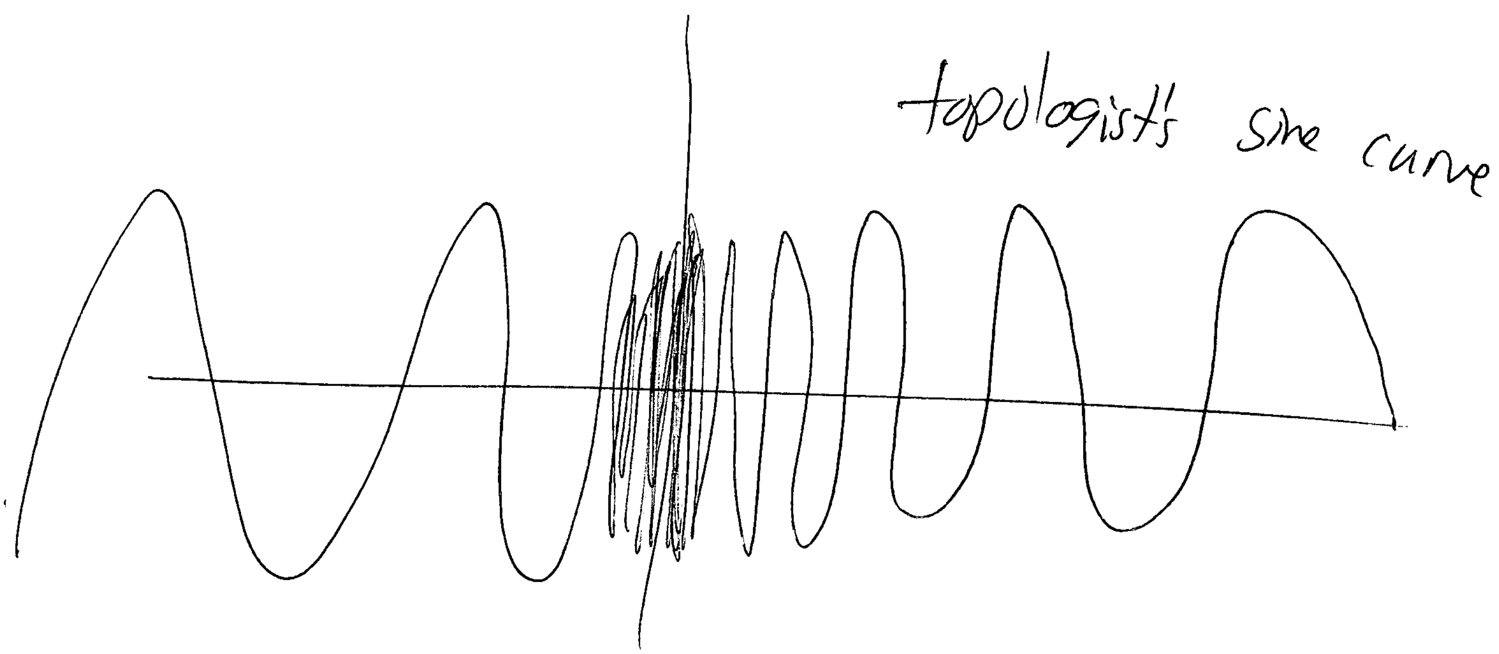


$f(x) = \sin(x)$

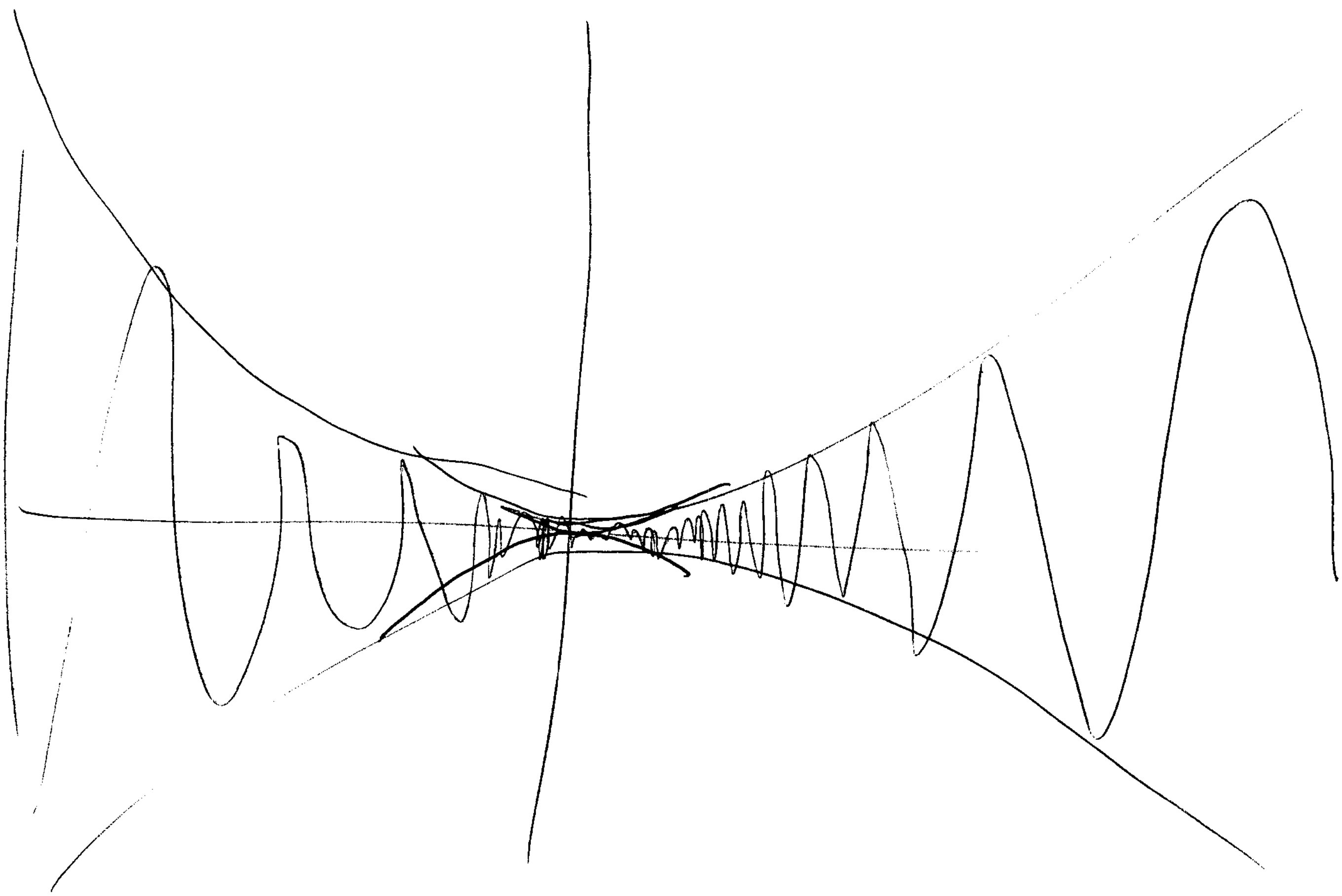


$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

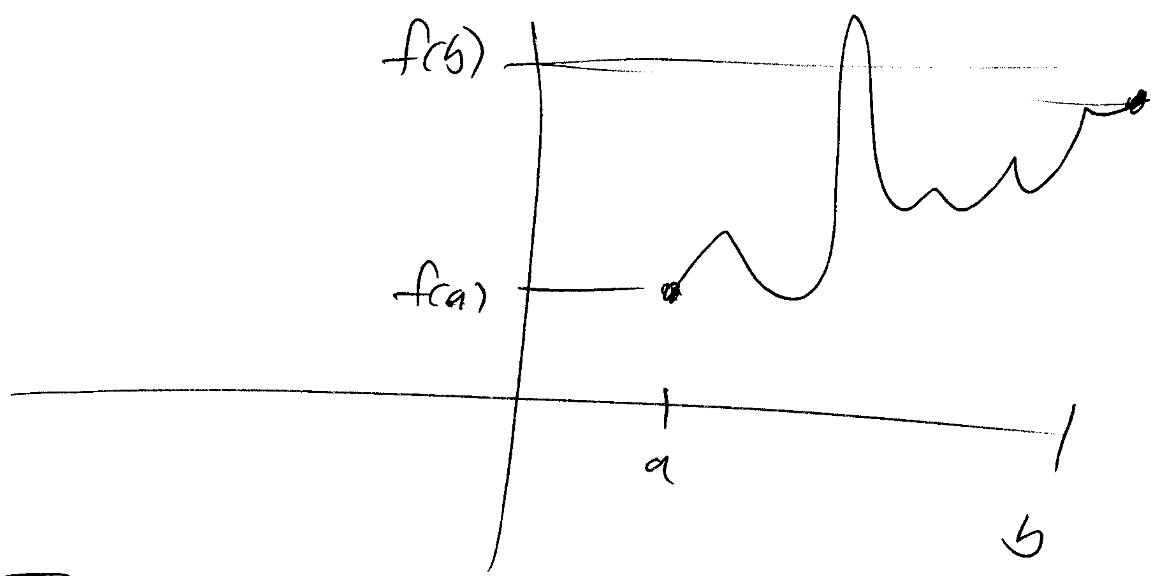
topologist's sine curve.



$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

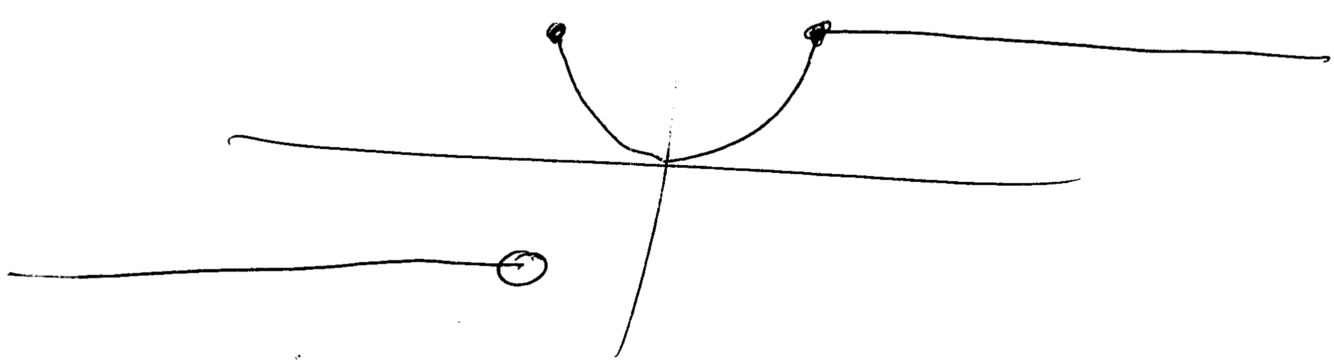


If $f(x)$ is continuous on $[a, b]$, then $f(x)$ takes on all values between $f(a)$ and $f(b)$.



Intermediate Value Theorem (IVT)

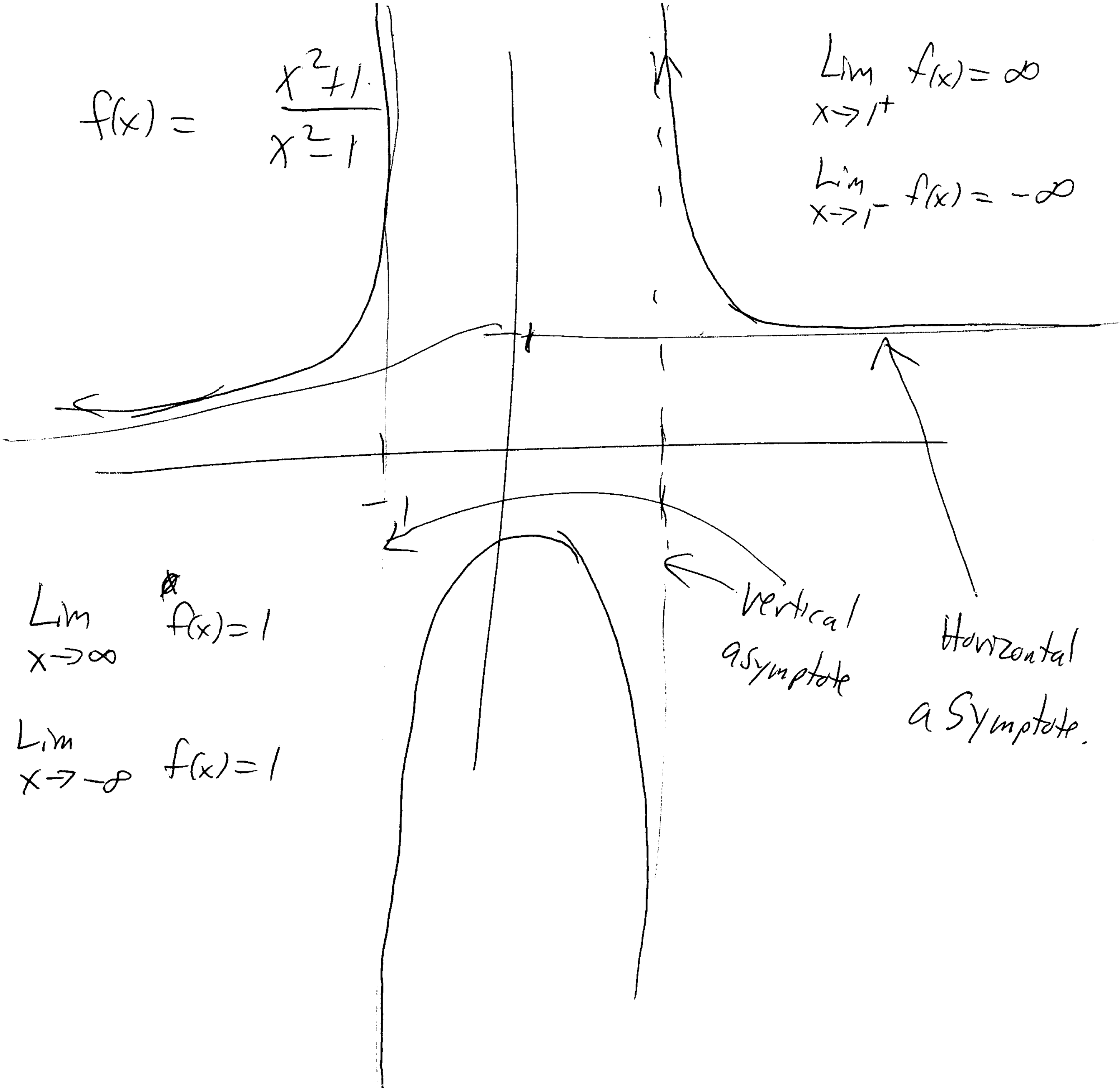
$$f(x) = \begin{cases} -1 & x < -1 \\ x^2 & -1 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$



$$f(x) = \frac{x^2+1}{x^2-1}$$

$\lim_{x \to 1^+} f(x) = \infty$

$\lim_{x \to 1^-} f(x) = -\infty$



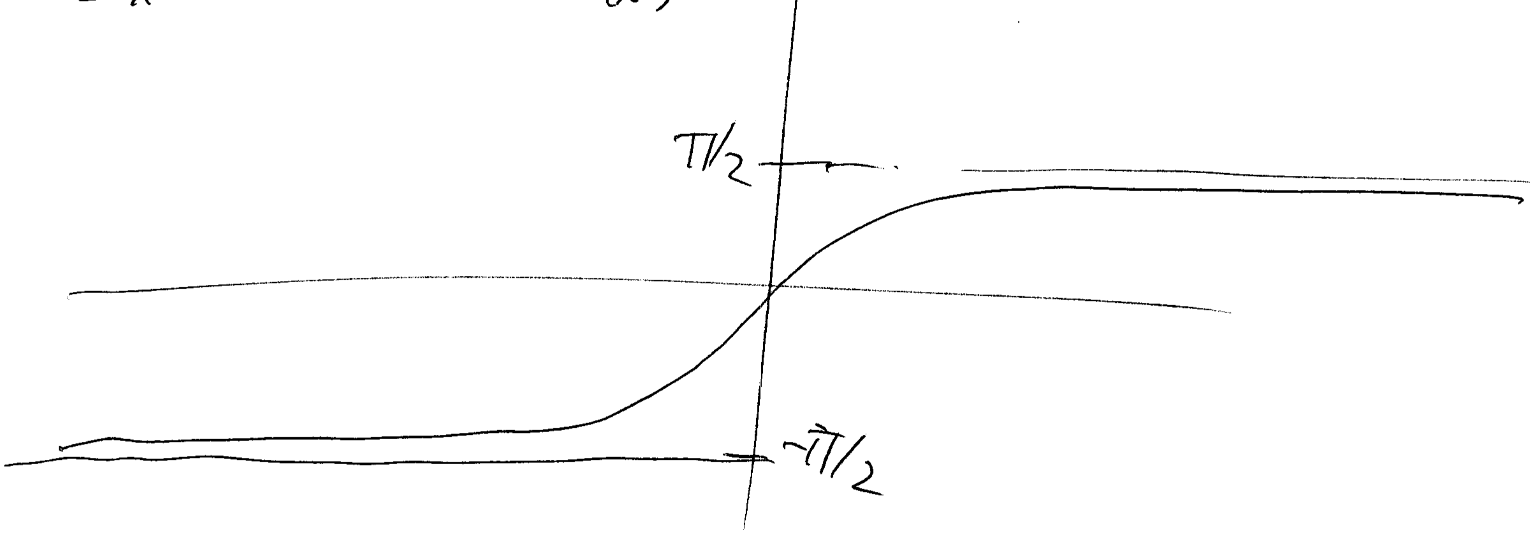
$\lim_{x \to \infty} f(x) = 1$

$\lim_{x \to -\infty} f(x) = 1$

If $\lim\limits_{x \to \infty} f(x) = a$, then $y = a$ is a horizontal asymptote.

If $\lim\limits_{x \to -\infty} f(x) = a$, 〃

Ex: $f(x) = \tan^{-1}(x)$



$$\lim_{x\to\infty} \frac{x^2+1}{x^2-1} = \lim_{x\to\infty} \frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}} = \frac{\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(1-\frac{1}{x^2}\right)}$$

$$= \frac{1}{1} = 1$$

$$\lim_{x\to\infty} \frac{\sqrt{x^2+x}}{x} = \lim_{x\to\infty} \sqrt{\frac{x^2+x}{x^2}}$$

$$= \lim_{x\to\infty} \sqrt{1+\frac{1}{x}} = \sqrt{\lim\left(1+\frac{1}{x}\right)} = \sqrt{1} = 1$$

When looking at ratios of polys or $\sqrt{\ }$, only leading term matters

$$\lim_{x\to+\infty} \frac{3x^4+17x^3+139x^2-100{,}000}{4x^4+x^2+1983}$$

$$= \lim_{x\to-\infty} \frac{3+\frac{17}{x}+\frac{139}{x^2}-\frac{100{,}000}{x^4}}{4+\frac{1}{x^2}+\frac{1983}{x^4}} = \frac{3}{4}$$

$$\lim_{x\to\infty} \frac{x^4+3x^3+17x^2+5x+9}{5x^3+18x-137}$$

$$= \lim_{x\to\infty} \frac{x+3+\frac{17}{x}+\frac{5}{x^2}+\frac{9}{x^3}}{5+\frac{18}{x^2}-\frac{137}{x^3}} = \infty$$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \begin{cases} 0 & \text{if } m>n \\ a_n/b_m & \text{if } m=n \\ \pm\infty & \text{if } m<n \end{cases}$$

$$\lim_{x\to -\infty} \frac{f(x)}{g(x)} =$$

$$\lim_{x \to \infty} \left(\sqrt{x^2+x} - x\right)\left(\frac{\sqrt{x^2+x} + x}{\sqrt{x^2+x} + x}\right)$$

$$= \lim_{x \to \infty} \frac{(x^2+x) - x^2}{\sqrt{x^2+x} + x} = \lim_{x \to \infty} \frac{x}{\sqrt{x^2+x} + x}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{1+\frac{1}{x}} + 1}$$

$$\left(\frac{\sqrt{x^2+x}}{x} = \sqrt{\frac{x^2+x}{x^2}} = \sqrt{1+\frac{1}{x}}\right)$$

$$= \frac{1}{2}$$

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$$

$$=\lim_{x\to\infty}\left(\frac{\left(\sqrt{x^2+x}-x\right)\left(\sqrt{x^2+x}+x\right)}{\left(\sqrt{x^2+x}+x\right)}\right)$$

$$=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x}=\lim_{x\to\infty}\frac{x/x}{\left(\sqrt{x^2+x}+x\right)/x}$$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{\frac{x^2+x}{x^2}}+\frac{x}{x}}=\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1}$$

$$=\frac{1}{\lim_{x\to 0}\left(\sqrt{1+\frac{1}{x}}+1\right)}=\frac{1}{\lim_{x\to\infty}\left(\sqrt{1+\frac{1}{x}}\right)+1}$$

$$=\frac{1}{\sqrt{\lim\left(1+\frac{1}{x}\right)}+1}=\frac{1}{\sqrt{1}+1}=\frac{1}{2}$$

If $f(x)$ is cont,

$$\lim_{x \to a} f(x) = f\left(\lim_{x \to a} x\right) = f(a)$$