On L’Hôpital’s Rule

There are three versions of L’Hôpital’s Rule, which I call “baby L’Hôpital’s rule”, “macho L’Hôpital’s rule” and “extended L’Hôpital’s rule”. The baby and macho versions refer to the problem of evaluating \( \lim_{x \to a} \frac{f(x)}{g(x)} \), where \( \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \). In other words, indeterminate forms of the type “0/0”, with \( a \) finite. (Also to limits as \( x \to a^+ \) and as \( x \to a^- \).) The extended form also applies to forms of the type \( \infty/\infty \) and to limits as \( x \to \pm \infty \).

1 The three theorems

Theorem 1 (Baby L’Hôpital’s Rule) Let \( f(x) \) and \( g(x) \) be continuous functions on an interval containing \( x = a \), with \( f(0) = g(0) = 0 \). Suppose that \( f \) and \( g \) are differentiable, and that \( f' \) and \( g' \) are continuous. Finally, suppose that \( g'(a) \neq 0 \). Then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}.
\]

Also,

\[
\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}
\]

and

\[
\lim_{x \to a^-} \frac{f(x)}{g(x)} = \lim_{x \to a^-} \frac{f'(x)}{g'(x)}.
\]

The baby version is easy to prove, and is good enough to compute limits like

\[
\lim_{x \to 0} \frac{\sin(2x)}{x + x^2}.
\]

However, it isn’t good enough to compute limits like

\[
\lim_{x \to 0} \frac{1 - \cos(2x)}{x^2},
\]

since in that case \( g'(0) = 0 \). To solve problems like (2), we need the macho version:
Theorem 2 (Macho L’Hôpital’s Rule) Suppose that $f$ and $g$ are continuous on a closed interval $[a, b]$, and are differentiable on the open interval $(a, b)$. Suppose that $g'(x)$ is never zero on $(a, b)$, and that \( \lim_{x \to a^+} f'(x)/g'(x) \) exists, and that \( \lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0 \). Then

\[
\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.
\]

Note that this theorem doesn’t require anything about $g'(a)$, just about how $g'$ behaves to the right of $a$. An analogous theorem applies to the limit as $x \to a^-$ (and requires $f$ and $g$ and $f'$ and $g'$ to be defined on an interval that ends at $a$, rather than one that starts at $a$). You can combine the two to get a theorem about an overall limit as $x \to a$.

The conclusion of Macho L’Hôpital’s Rule relates one limit (of $f/g$) to another limit (of $f'/g'$), and not to the value of $f'(a)/g'(a)$. This is what allows the theorem to be used recursively to solve problems like (2). Finally, we have the

Theorem 3 (Extended L’Hôpital’s Rule) L’Hôpital’s rule applies to indefinite forms of type “\( \infty/\infty \)” as well as “\( 0/0 \)”, and applies to limits as $x \to \pm \infty$ as well as to limits $x \to a^\pm$. In all of these cases,

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.
\]

2 Proofs of the baby and macho theorems

Suppose that $f(a) = g(a) = 0$ and $g'(a) \neq 0$. Then, for any $x$, $f(x) = f(x) - f(a)$ and $g(x) = g(x) - g(a)$. But then,

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{[f(x) - f(a)]/(x - a)}{[g(x) - g(a)]/(x - a)} = \lim_{x \to a} \frac{f(x)/f'(a)}{g(x)/g'(a)};
\]
since, by definition, 
\[ f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{and} \quad g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}. \]
Since \( f' \) and \( g' \) are assumed to be continuous, this is also
\[ \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{\lim_{x \to a} f'(x)}{\lim_{x \to a} g'(x)}. \]
That proves the baby version.

To prove the macho version, we first need a lemma:

**Theorem 4 (Souped up Mean Value Theorem)** If \( f(x) \) and \( g(x) \) are continuous on a closed interval \([a, b]\) and differentiable on the open interval \((a, b)\), then there is a point \( c \), between \( a \) and \( b \), where
\[ (f(b) - f(a))g'(c) = (g(b) - g(a))f'(c). \tag{3} \]
(When \( g(x) = x \), this is the same as the usual MVT.)

Proof of Souped up MVT: Consider the function
\[ h(x) = (f(x) - f(a))(g(b) - g(a)) - (f(b) - f(a))(g(x) - g(a)). \]
This is continuous on \([a, b]\) and differentiable on \((a, b)\), with
\[ h'(x) = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a)). \]
Note that \( h(a) = 0 = h(b) \). By Rolle’s Theorem, there a spot \( c \) where \( h'(c) = 0 \). But \( h'(c) = 0 \) is the same as equation (3).

Proof of Macho L’Hôpital’s Rule: By assumption, \( f \) and \( g \) are differentiable to the right of \( a \), and the limits of \( f \) and \( g \) as \( x \to a^+ \) are zero. Define \( f(a) \) to be zero, and likewise define \( g(a) = 0 \). Since these values agree with the limits, \( f \) and \( g \) are continuous on some half-open interval \([a, b)\) and differentiable on \((a, b)\).

For any \( x \in (a, b) \), we have that \( f \) and \( g \) are differentiable on \((a, x)\) and continuous on \([a, x]\). By the Souped up MVT, there is a point \( c \) between \( a \) and \( x \) such that \( f'(c)g(x) = f'(x)g(c) \). In other words, \( f'(c)/g'(c) = f(x)/g(x) \).

Also, as \( x \) approaches \( a \), \( c \) also approaches \( a \), since \( c \) is somewhere between \( x \) and \( a \). But then
\[ \lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(c)}{g'(c)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)}. \]
That last expression is the same as \( \lim_{x \to a^+} f'(x)/g'(x) \).
3 Proving the extended theorem

We’re going to use a single trick, over and over again. Namely, we can always rewrite $x$ as $1/(1/x)$, $f(x)$ as $1/(1/f(x))$ and $g(x)$ as $1/(1/g(x))$.

Suppose $L = \lim_{x \to a} \frac{f(x)}{g(x)}$, where both $f$ and $g$ go to $\infty$ (or $-\infty$) as $x \to a$. Also suppose that $L$ is neither 0 nor infinite. Then

$$L = \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{1}{f(x)} \frac{1}{g(x)}.$$ 

Since $1/g(x)$ and $1/f(x)$ go to zero as $x \to a$, we can apply the (baby or macho) L’Hôpital’s rule to this limit:

$$L = \lim_{x \to a} \frac{(1/g)'}{(1/f)'} = \lim_{x \to a} -\frac{g'(x)/g(x)^2}{f'(x)/f(x)^2} = \lim_{x \to a} \frac{f(x)^2 g'(x)}{g(x)^2 f'(x)} = \lim_{x \to a} \frac{f(x)^2}{g(x)^2} \lim_{x \to a} \frac{g'(x)}{f'(x)} = \frac{L^2}{\lim_{x \to a} [f'(x)/g'(x)]}.$$ 

Since $L = L^2 / \lim_{x \to a}[f'(x)/g'(x)]$, $L$ must equal $\lim_{x \to a}[f'(x)/g'(x)]$, which is what we wanted to prove.

This argument only works for finite and nonzero values of $L$. However, if $L = 0$, we can apply the same argument to the limit of $(f(x) + g(x))/g(x)$, which then does not equal zero. The upshot is that

$$1 + \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) + g(x)}{g(x)} = \lim_{x \to a} \frac{f'(x) + g'(x)}{g'(x)} = 1 + \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

hence that $\lim(f/g) = \lim(f'/g')$. Finally, if $\lim(f/g) = \pm \infty$, look instead at $\lim(g/f)$, which is then zero, so the previous reasoning applies. Since $0 = \lim(g/f) = \lim(g'/f')$, $\lim(f'/g')$ must be infinite. By the Souped up MVT, $f/g$ has the same sign as $f'/g'$, so we must have $\lim(f/g) = \lim(f'/g')$.

Now that we have L’Hôpital’s Rule for limits as $x \to a$ (or $x \to a^+$ or $x \to a^-$), we consider what happens as $x \to \infty$. Define a new variable
\[ t = \frac{1}{x}, \text{ so that } x \to \infty \text{ is the same as } t \to 0^+. \text{ Then} \]
\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{t \to 0^+} \frac{f(1/t)}{g(1/t)}. \]

But we know how to apply L'Hôpital's Rule to limits as \( t \to 0 \), so this turns into
\[ \lim_{t \to 0^+} \frac{\frac{d}{dt}f(1/t)}{\frac{d}{dt}g(1/t)} = \lim_{t \to 0^+} \frac{-f'(1/t)/t^2}{-g'(1/t)/t^2} = \lim_{t \to 0^+} \frac{f'(1/t)}{g'(1/t)}. \]

Converting back to \( x = 1/t \), we get
\[ \lim_{x \to \infty} \frac{f'(x)}{g'(x)}, \]

which is what we wanted. Computing a limit as \( x \to -\infty \) is similar, only with \( t \to 0^- \) instead of \( t \to 0^+ \).

That completes the proof of the Extended L'Hôpital's Rule.