1) (48 points, 2 pages) Compute the derivatives of the following functions with respect to $x$. Except in part (e), you do not need to simplify.

a) $x^2 \ln(x)$
   Apply the product rule to get $2x \ln(x) + x = x(1 + 2 \ln(x))$

b) $\tan^{-1}(x)$
   Apply the quotient rule to get
   $$\frac{\frac{1}{x^2+1} - 2x \tan^{-1}(x)}{(x^2+1)^2} = \frac{1 - 2x \tan^{-1}(x)}{(1 + x^2)^2}$$

c) $\sin^5(e^x + 7)$.
   Apply the chain rule several times in a row. The derivative is
   $$5 \sin^4(e^x + 7) \frac{d}{dx}(\sin(e^x + 7))$$
   $$= 5 \sin^4(e^x + 7) \cos(e^x + 7) \frac{d}{dx}(e^x + 7)$$
   $$= 5 \sin^4(e^x + 7) \cos(e^x + 7)e^x/(e^x + 7).$$

d) $\frac{e^{3x}}{x^2 + \cos(5x)}$
   Apply the quotient rule and then the chain rule (to get the derivatives of $e^{3x}$ and $\cos(5x)$). The answer is
   $$\frac{3e^{3x}(x^2 + \cos(5x)) - e^{3x}(2x - 5 \sin(5x))}{(x^2 + \cos(5x))^2}$$

e) $\sin^{-1}(\cos(x))$, with $0 < x < \pi/2$. Simplify your answer as much as possible!
   There are two ways to do this. The first is to notice that $\sin^{-1}(\cos(x)) = \frac{\pi}{2} - x$, whose derivative is $-1$. The second is to apply the chain rule to get
   $$\frac{1}{\sqrt{1-\cos^2(x)}}(-\sin(x)) = -\sin(x)/\sin(x) = -1.$$

f) $x^{(x^2)}$
   Use logarithmic differentiation. If $y = x^{x^2}$, then $\ln(y) = x^2 \ln(x)$, so
   $$y' = (\ln(y))' = (x + 2x \ln(x)) (as in part (a)), so y' = y(\ln(y))' = x^2(x + 2x \ln(x)) = x^{x^2+1}(1 + 2 \ln(x)).$$ Note: THERE IS NO POWER RULE for expressions of the form $f(x)^{g(x)}$! The power rule only applies to expressions like $u^n$, where $n$ is a constant.
2) The curve \( x^2 \ln(y) + ye^{x-3} = 1 \) goes through the point \( P = (3, 1) \). Find the equation of the line that is tangent to the curve at \( P \).

Taking a derivative of the equation with respect to \( x \) gives

\[
2x \ln(y) + \frac{x^2}{y} y' + ye^{x-3} + e^{x-3} y' = 0.
\]

Solving for \( y' \) gives

\[
y' = -\frac{(2x \ln(y) + ye^{x-3})}{\frac{x^2}{y} + e^{x-3}}.
\]

Plugging in \( x = 3 \) and \( y = 1 \) gives \( y' = -1/10 \). Since our tangent line has slope \(-1/10\) and goes through \((3, 1)\), its equation is \( y - 1 = -\frac{1}{10}(x - 3) \), or \( y = -\frac{x}{10} + \frac{13}{10} \).

3) Estimate \( \sqrt{628} \) and \( \sqrt{623} \), each to within .01. (Hint: Use the fact that \( \sqrt{625} = 25 \).)

Let \( f(x) = \sqrt{x} \), so \( f'(x) = 1/2\sqrt{x} \). Take \( a = 625 \), so \( f(a) = 25 \) and \( f'(a) = 1/50 \). Our tangent line is then \( y - 25 = (x - 625)/50 \), or \( y = 25 + (x - 625)/50 \). At \( x = 628 \) this gives \( y = 25.06 \), and at \( x = 623 \) it gives \( y = 24.96 \). Not only are these estimates of \( \sqrt{628} \) and \( \sqrt{623} \) accurate to .01, they are good to within .0001, since \( \sqrt{625} \approx 25.059928 \) and \( \sqrt{623} \approx 24.95996 \).

4) A F-15 fighter jet is flying 1 km above the ground, and will soon pass directly overhead. It is flying due east at 0.6 km/sec. Where will it be when the distance between you and the plane is decreasing at 0.3 km/sec? That is, how far west of you will the plane be? (Obviously it will still be a kilometer above the ground.) [In case you’re interested, here’s the physics behind the problem. A jet flying faster than sound generates a sonic boom in your direction when it is approaching you at exactly the speed of sound, which is a little over 0.3 km/s. If a jet flies by at Mach 2, it will take a few seconds for the boom to reach you, but the boom will come from exactly the spot that you calculate in this problem.]

Let \( x \) be the horizontal distance to the jet, and let \( r \) be the diagonal distance, both measured in kilometers. We then have \( r^2 = x^2 + 1 \), so \( 2r(dr/dt) = 2x(dx/dt) \). Since \( dr/dt = -0.3 \) and \( dx/dt = -0.6 \), we must have \( r = 2x \). But then \( (2x)^2 = 1 + x^2 \), so \( 3x^2 = 1 \), so \( x = \sqrt{3}/3 \) and the jet is \( \sqrt{3}/3 \) kilometers to the west. (The direction of the jet is 30 degrees to the west of vertical.)
5) Find all the critical points of the function \( f(x) = \frac{x^2}{1 + x^4} \). Then use these critical points to find the (global) maximum and minimum values of \( f(x) \) on the interval \([-10, 10]\).

\[
f'(x) = \frac{[(1 + x^4)2x - x^2(4x^3)]}{(1 + x^4)^2} = \frac{2x(1 - x^4)}{(1 + x^4)^2}. \]

This always exists, and is zero when \( x = 0 \) or \( x = \pm 1 \).

Since \( f(0) = 0 \), \( f(1) = f(-1) = 1/2 \) and \( f(10) = f(-10) = 100/10,001 \), the maximum value is \( 1/2 \), and is achieved at \( x = \pm 1 \), and the minimum value is 0 and is achieved at \( x = 0 \).

6) (12 points) The position of a particle at time \( t \) is given by the function \( f(t) = t^3 - 3t \). (a) What are the position, velocity and acceleration of the particle at time \( t = -2 \)?

The velocity function is \( f'(t) = 3t^2 - 3 \) and the acceleration is \( f''(t) = 6t \). Plugging in \( t = -2 \) we get position = -2, velocity = 9 and acceleration = -12.

b) Indicate all times when the particle is moving forwards (e.g., your answer might be something like “when \( t > -7 \)”).

The particle is moving forwards when \( |t| > 1 \), which is when \( f'(t) > 0 \). Between \( t = -1 \) and \( t = 1 \), the particle is moving backwards.

c) Indicate all times when the particle is accelerating forwards.

The particle is accelerating forwards when \( t > 0 \), since then \( f''(t) = 6t > 0 \).