M408N Final Exam Solutions, December 13, 2011

1) (32 points, 2 pages) Compute dy/dx in each of these situations. You do not need to simplify:

a) $y = x^3 + 2x^2 - 14x + 32$

$y' = 3x^2 + 4x - 14$, by the nx^{n-1} formula.

b) $y = (x^3 + 7)^5$.

$y' = 5(x^3 + 7)^4(3x^2) = 15x^2(x^3 + 7)^4$ by the chain rule.

c) $y = \frac{\sin(x)}{x^2+1}$

$y' = \frac{(x^2+1)\cos(x)-2x\sin(x)}{(x^2+1)^2}$ by the quotient rule.

d) $y = \ln(\sin(x^2))$

$y' = \frac{2x\cos(x^2)}{\sin(x^2)}$ by the chain rule applied twice. Once to $\ln(u)$ and once to $\sin(u)$.

e) $y = e^x \tan^{-1}(x)$

$y' = e^x \tan^{-1}(x) + \frac{e^x}{1+x^2}$ by the product rule.

f) $y = x^{\sin(x)}$

Use logarithmic differentiation. Since $\ln(y) = \sin(x)\ln(x)$, we have $y'/y = (\ln(y))' = \cos(x)\ln(x) + \frac{\sin(x)}{x}$, so $y' = x^{\sin(x)}\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right)$.

g) $xy + e^x + \ln(y) = 17$. (For this part, you can leave your answer in terms of both x and y)

Use implicit differentiation. $y + xy' + e^x + y'/y = 0$, so $(x + \frac{1}{y})y' = -(e^x + y)$, so $y' = \frac{-(e^x+y)}{x+(1/y)}$.

h) $y = \int_3^{2x} \sin(t^2)dt$.

This is the fundamental theorem of calculus (version 1) combined with the chain rule. If $u = 2x$, then $dy/du = \sin(u^2) = \sin(4x^2)$, so $dy/dx = (dy/du)(du/dx) = 2\sin(4x^2)$.

2) (8 points) Some values of the function differentiable $f(x)$ are listed in the following table.

x	$f(x)$
2.95	8.8050
2.96	8.8432
2.97	8.8818
2.98	8.9208
2.99	8.9602
3.00	9.0000
3.01	9.0402
3.02	9.0808
3.03	9.1218
3.04	9.1632
3.05	9.2050

a) Compute the average rate of change between $x = 2.99$ and $x = 3.04$.

$\Delta y/\Delta x = (f(3.04) - f(2.99))/.05 = .2030/.05 = 4.06$.

b) Estimate, as accurately as you can, the value of $f'(3)$.

Add a couple more columns to the table:

x	$f(x)$	$f(x) - f(3)$	$(f(x) - f(3))/(x - 3)$
2.95	8.8050	-.195	3.90
2.96	8.8432	-.1568	3.92
2.97	8.8818	-.1172	3.94
2.98	8.9208	-.0792	3.96
2.99	8.9602	-.0398	3.98
3.00	9.0000	0	???
3.01	9.0402	.0402	4.02
3.02	9.0808	.0808	4.04
3.03	9.1218	.1218	4.06
3.04	9.1632	.1632	4.08
3.05	9.2050	.2050	4.10

The limit, as $x \to 3$, of $(f(x) - f(3))/(x - 3)$ sure looks like it's 4, so we estimate that $f'(3) = 4$. (Those who used a single estimate using $h = .01$ or $h = -.01$ and got 3.98 or 4.02 get full credit, but 4 is more accurate.)

3) (10 points) Let $f(x) = 10x^3 - 74$.

a) Find the equation of the line tangent to the curve $y = f(x)$ at (2,6).

Since $f'(x) = 30x^2$, $f'(2) = 120$, so our tangent line is $y - 6 = 120(x - 2)$. You can expand this out to $y = 120x - 234$, but it's a LOT simpler to work with $y = 6 + 120(x - 2)$.

b) Use this tangent line (or equivalently, a linear approximation) to estimate $f(2.1)$.

If $x = 2.1$, then $6 + 120(x - 2) = 6 + 12 = 18$, so $f(2.1) \approx 18$. (In reality, it's around 18.61).

c) Use this tangent line (or equivalently, a linear approximation) to estimate a value of x for which $f(x) = 0$. (Congratulations. You just computed the cube root of 7.4 by hand.)

If $6 + 120(x - 2) = 0$, then $(x - 2) = -6/120 = -.05$, so $x = 1.95$. This calculation is the same thing as applying one step of Newton's method.

4) (12 points, 2 pages!) The position of a particle is $f(t) = t^4 - 6t^2 + 8$. (Note that this factors as $(t^2 - 2)(t^2 - 4)$. Note also that t can be positive or negative; the domain of the function is the entire real line.)

a) Make a sign chart for f, indicating the values of t where $f(t)$ is positive, where $f(t)$ is negative, and where $f(t) = 0$.

Since $f(t) = (t^2 - 2)(t^2 - 4)$, $f(t)$ will be positive when $t^2 < 2$ or $t^2 > 4$, and negative when $2 < t^2 < 4$. In other words, $f(t)$ is positive on $(-\infty, -2)$, negative on $(-2, -\sqrt{2})$, positive on $(-\sqrt{2}, \sqrt{2})$, negative on $(\sqrt{2}, 2)$, and positive on $(2, \infty)$, and equals zero at $t = \pm\sqrt{2}$ and $t = \pm 2$.

b) At what times is the particle moving forwards? (Either express your answer in interval notation or make a relevant sign chart.)

The velocity if $f'(t) = 4t^3 - 12t = 4t(t^2 - 3)$. The particle is moving forwards when $t > 0$ and $t^2 > 3$ or when $t < 0$ and $t^2 < 3$. In other words, it's moving forwards on the intervals $(\sqrt{3}, \infty)$ and $(-\sqrt{3}, 0)$.

c) At what times is the velocity increasing?

The acceleration is $f''(t) = 12t^2 - 12 = 12(t^2 - 1)$. The velocity is increasing when this is positive, namely when $t < -1$ or $t > 1$.

d) Sketch the graph $y = f(t)$. Mark carefully the local maxima, the local minima, and the points of inflection.

The graph looks like a curvy W, and is sometimes called a "Mexican hat" function. There is a local maximum at the point $(0, 8)$, local minima at $(\pm\sqrt{3}, -1)$, and points of inflection at $(\pm 1, 3)$, and crosses the x-axis at $x = \pm 2$ and $\pm\sqrt{2}$.

5) (8 points) Consider the function

$$f(x) = \begin{cases} 1 + x & x \leq 0 \\ e^x & x > 0. \end{cases}$$

a) Compute

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x}.$$

Since $f(x) = e^x$ for $x > 0$, and since $f(0) = 1$, this is $\lim_{x \to 0^+} \frac{e^x - 1}{x}$, which is (by definition!) the derivative of e^x at $x = 0$. This equals 1. (You can also get the answer from L'Hôpital's rule).

b) Compute

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x}.$$

Since $f(x) = 1 + x$ for $x < 0$, this is $\lim_{x \to 0^-} \frac{x}{x} = 1$.

c) Is f differentiable at $x = 0$? Explain why or why not. If f is differentiable, compute $f'(0)$.

Yes, it's differentiable. Since the limits of $\frac{f(x) - f(0)}{x}$ from the two directions are the same, there is an overall limit, namely 1. By definition, $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = 1$. Some people argued that $f(x)$ was differentiable since the limits of $f(x)$ from the two directions are the same (and equal $f(0)$). That only makes the function *continuous*, not differentiable.

6) (6 points) Consider the expression

$$\lim_{N \to \infty} \left(\sum_{j=1}^{N} \frac{6}{N} \left(1 + \frac{2j}{N} \right)^2 \right)$$

a) Rewrite this expression as a definite integral.

There are several ways to write this as an integral. The simplest is $\int_1^3 3x^2 dx$. Taking $a = 1$ and $b = 3$, we have $\Delta x = 2/N$, $x_j = 1 + 2j/N$, and our sum is $\sum_{j=1}^{n} 3x_j^2 \Delta x$, whose limit is $\int_1^3 3x^2 dx$.

Note that $(1+\frac{2j}{N})$ ranges from $1+2/N \approx 1$ to $1+2N/N = 3$, so if we're integrating something times x^2, the integral must be from 1 to 3. However, there are other ways to write the integral. If we take $a = 0$ and $b = 2$, we get $\int_0^2 3(1+x)^2 dx$. If we take $a = 0$ and $b = 1$, we get $\int_0^1 6(1+2x)^2 dx$. If we take $a = 0$ and $b = 6$, we get $\int_0^6 (1+\frac{x}{3})^2 dx$. However you slice it, the quantity being squared goes from 1 to 3.

b) Evaluate this integral (using the Fundamental Theorem of Calculus).

$\int_1^3 3x^2 dx = x^3|_1^3 = 3^3 - 1^3 = 26.$

7) (8 pts) Cops and robbers.

A person is mugged at the corner of a north-south street and an east-west street, and calls for help. A little while later, at time $t = 0$, the robber is 200m east of the intersection, running east with a speed of 5m/s. (m means "meter" and s means "second") At the same time, a cop is 500m north of the intersection, running south at 5m/s.

a) At what rate is the (straight-line) distance between the cop and the robber changing at $t = 20$ seconds?

Let $x = 200 + 5t$ be the position of the robber (east of the intersection) and let $y = 500 - 5t$ be the position of the cop (north of the intersection). Note that $dx/dt = 5$ is positive and $dy/dt = -5$ is negative, since the robber is running *away* from the intersection and the cop is running *toward* it. The distance r between them satisfies $r^2 = x^2 + y^2$, so

$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 10(x-y),$$

so $dr/dt = 5(x-y)/r$. At time $t = 20$, we have $x = 300$ and $y = 400$, hence $r = 500$, so $dr/dt = 5(-100)/500 = -1m/s$.

b) At what time is the distance between the cop and the robber minimized? (You can restrict your attention to the interval $0 \leq t \leq 100$ seconds.)

Notice that r first decreases (when $x < y$) and then increases (when $x > y$), so the critical point (where $x = y$) must be a minimum. This occurs when $0 = x - y = 10t - 300$, so $t = 30$. (At this time, $x = y = 350$ and $r = 350\sqrt{2}$. You weren't actually asked for the minimum distance, but there it is.)

8. (8 points) Compute (with justification) the two limits

a)

$$\lim_{x\to 1} \frac{\sin(\pi x)}{\ln(x)}$$

This is a 0/0 indeterminate form, so use L'Hôpital's rule: $\lim_{x\to 1} \frac{\sin(\pi x)}{\ln(x)} = \lim_{x\to 1} \frac{\pi \cos(\pi x)}{1/x} = \frac{\pi \cos(\pi)}{1} = -\pi$.

b)

$$\lim_{x\to\infty} \frac{\sin(x^2)}{x}.$$

This is *not* an indeterminate form, and L'Hôpital's rule does *not* apply. The numerator is oscillates between −1 and 1, while the denominator gets bigger and bigger. The limit of the ratio is zero. (More precisely, $-1/x \leq \frac{\sin(x^2)}{x} \leq 1/x$. Since $-1/x$ and $1/x$ both go to zero, the sandwich theorem says that $\sin(x^2)/x$ must also go to zero.)

9) (8 points) The acceleration of a particle is given by the function $a(t) = 4\cos(t) - 6t + 6$.

a) At $t = 0$, the velocity is $v(0) = 1$. Find the velocity $v(t)$ as a function of time.

The velocity is the anti-derivative of the acceleration, so $v(t) = 4\sin(t) - 3t^2 + 6t + C$ for some constant C. Note that the anti-derivative of $4\cos(t)$ is $4\sin(t)$, not $2\sin^2(t)$! Plugging in $v(0) = 1$ gives $C = 1$, so $v(t) = 4\sin(t) - 3t^2 + 6t + 1$.

b) Let $x(t)$ denote the position at time t. Compute $x(2) - x(0)$. (There is more than one way to do this, but there is only one right answer.)

We can do this either with anti-derivatives or with integrals. It's really all the same, thanks to the Fundamental Theorem of Calculus.

Using antiderivatives, we compute $x(t) = -4\cos(t) - t^3 + 3t^2 + t + D$ for some *unknown* constant D. Computing $x(2) - x(0)$ we get $-4\cos(2) - 8 + 12 + 2 + D - (-4 + D) = 10 - 4\cos(2)$. The constant D has conveniently disappeared. If you prefer integrals, we compute $x(2) - x(0) = \int_0^2 v(t)dt = (-4\cos(t) - t^3 + 3t^2 + t)|_0^2 = 10 - 4\cos(2)$. When integrating, we don't have to worry about the constant D, since *any* anti-derivative will work for the FTC. A shocking number of people didn't remember that $\sin(0) = 0$, $\cos(0) = 1$, and that that $\cos(2)$ is a complicated number that *isn't* 0 or 1! ($\cos(2) \approx -0.416$).