1) (15 pts) Suppose that at a certain time there are 500 bacteria growing in a Petri dish. The population grows exponentially, doubling every hour.

a) Find a formula for the number \( x(t) \) of bacteria \( t \) hours later.

\[ x(t) = 500 \cdot 2^t, \] since we start with 500 and double every hour.

b) Find a formula for \( t \) in terms of \( x \).

\[ 2^t = x/500, \text{ so } t = \log_2(x/500). \] You could also write \( \log_2(x) - \log_2(500) \) or \( \log_2(x) - 3 \log_2(5) - 2 \).

c) Now suppose that the bacteria double every 20 minutes (instead of every hour). How does this change the answers to parts (a) and (b)?

In that case we would have \( x(t) = 500 \cdot 2^{3t} = 500 \cdot 8^t \) and \( t = \frac{\log_2(x/500)}{3} \) (or \( \log_8(x/500) \)).

2. (15 points) Compute the following quantities exactly. The answers may involve square roots, in which case you can leave your answers looking like \( \sqrt{3}/7 \) or \( 5\sqrt{2} \) (which aren’t actually the answers, of course).

a) Draw a right triangle where one of the angles has a tangent of 2. Mark the lengths of the three sides clearly. Then compute the sine of that angle.

The triangle I had in mind has opposite side of length 2 and adjacent side of length 1 (that is, vertices at (0,0), (1,0) and (1,2)). The hypotenuse is then \( \sqrt{2^2 + 1^2} = \sqrt{5} \), and the sine is \( \text{opposite/hypotenuse} = 2/\sqrt{5} \) (or \( 2\sqrt{5}/5 \)).

b) Now draw a right triangle involving an angle whose sine is 1/2. Mark the lengths of all three sides, and compute the secant of that angle.

Now the opposite side has height 1/2 and the hypotenuse has length 1, so the adjacent side has length \( \sqrt{1 - (1/2)^2} = \sqrt{3}/2 \). The secant is hypotenuse/adjacent = \( 2/\sqrt{3} \).

c) Compute \( \sin(\cos^{-1}(1/3)) \).

This is the sine of an angle whose cosine is 1/3. Draw a triangle with adjacent side 1/3 and hypotenuse 1, hence opposite side of length \( \sqrt{8}/9 \), so the sine of the angle is \( \sqrt{8}/9 = 2\sqrt{2}/3 \).
3. (10 pts) Consider the function $f(x) =\begin{cases} 2x + 1 & x < 2 \\ 4 & x = 2 \\ 7 - x & x > 2 \end{cases}$

a) Compute $\lim_{{x \to 2^+}} f(x)$, $\lim_{{x \to 2^-}} f(x)$ and $\lim_{{x \to 2}} f(x)$, if they exist.

$\lim_{{x \to 2^+}} f(x) = 7 - x = 5$, $\lim_{{x \to 2^-}} f(x) = \lim_{{x \to 2^-}} 2x + 1 = 5$.

Since both 1-sided limits give 5, $\lim_{{x \to 2}} f(x) = 5$.

b) Is $f(x)$ continuous everywhere? Why or why not?

$f(x)$ is NOT continuous, since $5 = \lim_{{x \to 2}} f(x) \neq f(2) = 4$. The function has a removable discontinuity at $x = 2$.

4. (30 pts) Compute the following limits:

a) $\lim_{{x \to -1}} \frac{x^2 - x - 2}{x - 2}$.

Both the numerator and denominator are continuous, and the denominator doesn’t go to 0 as $x \to -1$, so this function is continuous at $x = -1$. Plug in $x = -1$ to get $0/(-3) = 0$.

b) $\lim_{{x \to 2}} \frac{x^2 - x - 2}{x - 2}$.

The numerator is $(x - 2)(x + 1)$, so we have $\lim_{{x \to 2}} \frac{(x-2)(x+1)}{x-2} = \lim_{{x \to 2}} x + 1 = 3$.

c) $\lim_{{x \to 1^+}} \frac{1}{1 - x}$.

When $x$ is slightly bigger than 1, the denominator is small and negative, while the numerator is close to 1, so the ratio is huge and negative. This makes the limit $-\infty$.

d) $\lim_{{x \to (\frac{\pi}{2})^+}} \sin(x) \tan(x)$.

First, expand $\sin(x) \tan(x) = \sin^2(x)/\cos(x)$. When $x$ is a little bigger than $\pi/2$, $\cos(x)$ is slightly negative, while $\sin^2(x)$ is approximately 1. This is just like (c), with a limit of $-\infty$.

e) $\lim_{{x \to \infty}} \frac{2x^3 - x^2 + 17x - 5}{3x^3 + 139x^2 - 47x + \pi}$

Divide the top and the bottom by $x^3$ to get $\lim_{{x \to \infty}} \frac{2 - x^{-1} + 17x^{-2} - 5x^{-3}}{3 + 139x^{-3} - 47x^{-2} + \pi x^{-3}} = 2/3$.

f) $\lim_{{x \to -\infty}} \frac{|x^5 + 3|}{x^4 + 2x^2 + 1}$.

The numerator grows faster than the denominator, so the limit is either
±∞. To see which, imagine plugging in a large negative value of \( x \). The numerator is positive (being an absolute value), as is the denominator (which is \( x^4 \) plus change), so the ratio is positive and growing. The limit is \( \infty \).

5. (30 points) True or False (no partial credit, and no penalty for guessing)

a) If \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to a^+} f(x) \) both exist, then \( \lim_{x \to a} f(x) \) exists.

   FALSE. The one-sided limits might be different. (E.g., \( \lim_{x \to 0} |x|/x \).)

b) If \( f(x) \) is a polynomial, then \( \lim_{x \to a} f(x) = f(a) \).

   TRUE. All polynomials are continuous.

c) The statement \( \lim_{x \to \infty} f(x) = -\infty \) means that whenever \( x \) is sufficiently large and positive, \( f(x) \) is large and negative.

   TRUE. (The precise definition requires us to say what “large” means.)

d) If \( f(x) \) and \( g(x) \) are continuous at \( x = a \), then so are \( f(x) + g(x) \), \( f(x)g(x) \), and \( f(x)/g(x) \).

   FALSE. For \( f(x)/g(x) \) to be continuous, or even defined, we also need \( g(a) \neq 0 \).

e) \( \ln(75e^2) - 2 \ln(5) - \ln(3) = 2 \).

   TRUE. \( \ln(75e^2) = \ln(3 \cdot 5^2 \cdot e^2) = \ln(3) + 2 \ln(5) + 2 \).

f) The inverse function of \( f(x) = 3e^x + 1 \) is \( f^{-1}(x) = \log_e \left( \frac{x-1}{3} \right) \).

   TRUE. If \( y = 3e^x + 1 \), then \( e^x = (y - 1)/3 \) and \( x = \ln((y - 1)/3) \). (And \( \ln = \log_e \))

g) For every \( x \) where both sides are defined, \( \cot^2(x) + 1 = \sec^2(x) \).

   FALSE. \( \cot^2(x) + 1 = \csc^2(x) \), not \( \sec^2(x) \).

h) If \( f(x) \) is continuous on the interval \([0, 4]\), then \( \lim_{x \to 3} f(x) \) must exist.

   TRUE. Not only that, but the limit must equal \( f(3) \).

i) \( \log_{10}(e) = \log_e(10) \).

   FALSE. These numbers are reciprocals. Since \( 10 > e \), \( \log_e(10) > 1 \) while \( \log_{10}(e) < 1 \).

j) \( \log_{32}(2) = 1/5 \).

   TRUE. \( 32 = 2^5 \), so \( 2 = 32^{1/5} \).