M346 First Midterm Exam, February 11, 2009

1a) In \( \mathbb{R}^3 \), let \( E \) be the standard basis and let \( B = \{(1, 2, 3), (0, 1, 4), (0, 0, 1)\} \) be an alternate basis. Let \( v = \begin{pmatrix} 3 \\ -2 \\ 14 \end{pmatrix} \). Find \( P_{EB}, P_{BE} \) and \( [v]_B \).

1b) In \( \mathbb{R}_2[t] \), let \( E = \{1, t, t^2\} \) be the standard basis and let \( B = \{1 + 2t + 3t^2, t + 4t^2, t^2\} \) be an alternate basis, and let \( v = 3 - 2t + 14t^2 \). Find \( P_{EB}, P_{BE} \) and \( [v]_B \).

2. Let \( L : \mathbb{R}^3 \to \mathbb{R}^4 \) be given by \( L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ x_1 + 2x_2 + 3x_3 \\ x_1 + 3x_2 + 5x_3 \\ x_1 + 4x_2 + 7x_3 \end{pmatrix} \).

a) Find the matrix of \( L \) (relative to the standard bases for \( \mathbb{R}^3 \) and \( \mathbb{R}^4 \).

b) Let \( V = \{x \in \mathbb{R}^3 : L(x) = 0\} \). What is the dimension of \( V \)? Find a basis for \( V \).

3. On \( \mathbb{R}^2 \), consider the basis \( B = \{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\} \). Let \( L(x) = Ax \), where \( A = \begin{pmatrix} 19 & -30 \\ 10 & -16 \end{pmatrix} \).

a) Find the change-of-basis matrices \( P_{EB} \) and \( P_{BE} \), where \( E = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\} \) is the standard basis.

b) Find the matrix of \( L \) relative to the \( B \) basis.

c) If we were given the problem of solving the evolution equations \( x(n+1) = Ax(n) \), we would switch to coordinates \( y = [x]_B \). Rewrite the equations in terms of the variables \( y_1 \) and \( y_2 \). You do not need to solve these equations for \( y(n) \) in terms of \( y(0) \). Just get \( y(n+1) \) in terms of \( y(n) \).

4. True or False? Each question is worth 5 points. You do NOT need to justify your answers, and partial credit will NOT be given.

a) If four vectors in \( \mathbb{R}_3[t] \) are linearly independent, then they form a basis for \( \mathbb{R}_3[t] \).

b) If \( A \) is a \( 3 \times 5 \) matrix whose rank is two, then the set of solutions to \( Ax = 0 \) is a 2-dimensional subspace of \( \mathbb{R}^5 \).

c) If \( B \) and \( D \) are basis for a vector space \( V \), then the change-of-basis matrices
$P_{BD}$ and $P_{DB}$ are inverses of one another.

d) If $P_{BD}$ is a change-of-basis matrix, then for any vector $\mathbf{v}$, $[\mathbf{v}]_B = P_{DB} [\mathbf{v}]_D$.

e) The columns of an $m \times n$ matrix $A$ span $\mathbb{R}^m$ if and only the reduced row-echelon form of $A$ has a pivot in each column.