1a) Find a matrix with eigenvalues 3 and 4 and eigenvectors \((\begin{pmatrix} 3 \\ 2 \end{pmatrix})\) and \((\begin{pmatrix} 4 \\ 3 \end{pmatrix})\), respectively.

\[ A = PDP^{-1}, \text{ where } P = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \text{ and } D = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}. \]

Since the determinant of \(P\) is 1, we compute \(P^{-1} = \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}\) and multiply things out to get \(A = \begin{pmatrix} -5 & 12 \\ -6 & 12 \end{pmatrix}\).

b) Find a matrix with eigenvalues \(3 \pm 2i\) and eigenvectors \(\begin{pmatrix} 1 \\ \pm i \end{pmatrix}\).

\[ A = PDP^{-1}, \text{ where } P = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \text{ and } D = \begin{pmatrix} 3 + 2i & 0 \\ 0 & 3 - 2i \end{pmatrix}. \]

Since the determinant of \(P\) is \(-2i\), we compute \(P^{-1} = \frac{i}{2} \begin{pmatrix} -i & 1 \\ -i & 1 \end{pmatrix}\) and multiply things out to get \(A = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}\).

2. Find the eigenvalues of

\[
\begin{pmatrix}
3 & 5 & 7 & 6 \\
0 & 3 & 1 & 0 \\
0 & 5 & -1 & 0 \\
0 & 4 & 3 & 15
\end{pmatrix}
\]

You do not have to find the eigenvectors.

The matrix is block-triangular, with a northwest \(1 \times 1\) block and a southeast \(3 \times 3\) block. The \(3 \times 3\) block is itself block triangular. So we’re left with 3, 15, and the eigenvalues of \((\begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix})\). The rows add up to 4, and the trace is 2, so the eigenvalues of the \(2 \times 2\) block must be 4 and -2. The eigenvalues of the whole matrix are \(\{3, 4, -2, 15\}\).

3. The matrix \(A = \begin{pmatrix} 3 & 3 & -2.1 \\ -2.1 & -2.3 \end{pmatrix}\) has eigenvalues -3 and 4, with eigenvectors \((\begin{pmatrix} 1 \\ 3 \end{pmatrix})\) and \((\begin{pmatrix} -3 \\ 1 \end{pmatrix})\). We wish to solve \(x(n + 1) = Ax(n)\). As usual, let \(y = [x]_B\), where \(B\) is the basis of eigenvectors.

a) If \(x(0) = \begin{pmatrix} 10 \\ 20 \end{pmatrix}\), what is \(y(0)\)?

\[ P = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}, P^{-1} = \frac{1}{16} \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}, \text{ and } y(0) = P^{-1}x(0) = \begin{pmatrix} 7 \\ -1 \end{pmatrix}. \]
should check that $x(0) = 7b_1 - b_2$.

b) Find $y(n)$ for all $n \geq 0$.

Just multiply each coefficient by $\lambda^n$. $y(n) = \begin{pmatrix} 7 \cdot (-3)^n \\ -1 \cdot 4^n \end{pmatrix}$.

c) Find $x(n)$ for all $n \geq 0$.

$$x(n) = P y(n) = \begin{pmatrix} 7 \cdot (-3)^n + 3 \cdot 4^n \\ 21 \cdot (-3)^n - 4^n \end{pmatrix}.$$ 

4. a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & 3 \\ 2 & 8 \end{pmatrix}$.

Double-check that your eigenvectors are correct, as you will need them for the other parts!

The trace is 11 and the determinant is 18, so the eigenvalues are 9 and 2, with eigenvectors (obtained by row reduction) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

b) If $dx/dt = Ax$ and $x(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, what is the limit, as $t \to \infty$, of $x_1(t)/x_2(t)$? [Note: you do not have to actually compute $x(t)$ to do this!]

Since $x(0)$ is not an eigenvector, it is of the form $c_1 b_1 + c_2 b_2$, with $c_1$ and $c_2$ nonzero. Then $x(t) = c_1 e^{9t} b_1 + c_2 e^{2t} b_2$, which asymptotically points in the $b_1$ direction. So our limit is $1/2$.

Extra credit. If $x(n + 1) = Ax(n)$ and $x(0) = \begin{pmatrix} 3.14159 \\ 2.71828 \end{pmatrix}$, what is the limit, as $n \to \infty$, of $x_1(n + 1)/x_1(n)$? [For heaven’s sake, don’t attempt to do arithmetic with these numbers! Think about the long-term behavior we discussed on Wednesday.]

In this example, $x(n) = c_1 9^n b_1 + c_2 2^n b_2$ for some ugly numbers $c_1$ and $c_2$. The first term dominates, and it just gets multiplied by 9 each time, so our limit is 9.