1. Is it fixed yet? Consider the system of nonlinear differential equations:
\[
\begin{align*}
\frac{dx_1}{dt} &= x_1(3 - 2x_1 - x_2), \\
\frac{dx_2}{dt} &= x_2(5 - 2x_1 - 3x_2).
\end{align*}
\]

a) Find all the fixed points.
(0,0), (0, \frac{5}{3}), \left(\frac{3}{2}, 0\right) and (1,1)
b) For each fixed point, indicate how many stable modes, and how many unstable modes, there are.

Our derivative matrix is 
\[
\begin{pmatrix}
3 - 4x_1 - x_2 & -x_1 \\
-3x_2 & 5 - 5x_2 - 3x_1
\end{pmatrix}
\]

At (0,0) this is \( \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \) which has two positive eigenvalues, so there are two unstable modes and no stable modes. This is a source.

At \( (0, \frac{5}{3}) \) this is \( \begin{pmatrix} \frac{4}{3} & 0 \\ -5 & -5 \end{pmatrix} \) which has one positive eigenvalue, so there is one unstable mode and one stable mode. This is a saddle.

At \( \left(\frac{3}{2}, 0\right) \) this is \( \begin{pmatrix} -3 & -\frac{3}{2} \\ 0 & -\frac{1}{2} \end{pmatrix} \) which has one positive eigenvalue, so there is one unstable mode and one stable mode. This is a saddle.

At (1,1) this is \( \begin{pmatrix} -2 & -1 \\ -3 & -4 \end{pmatrix} \) which has two negative eigenvalues (-5 and -1), so there are two stable modes and no unstable modes. This is a sink.

2. Gram crackers. In \( \mathbb{R}^3 \), with the standard inner product, consider the vectors \( x_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \), \( x_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \), \( x_3 = \begin{pmatrix} 16 \\ 9 \\ -2 \end{pmatrix} \) that form a basis for \( \mathbb{R}^3 \).

a) Use the Gram-Schmidt process to convert this basis to an orthogonal basis \( \{y_1, y_2, y_3\} \). (Note: the vectors \( y_i \) do not have to be orthonormal, just orthogonal.)
\[
y_1 = x_1 = (1, 2, 3)^T.
\]
\[
y_2 = x_2 - \frac{14}{11}y_1 = (4, -1, 4)^T.
\]
\[
y_3 = x_3 - \frac{28}{11}y_1 - \frac{19}{11}y_2 = (11, 8, -9)^T.
\]
b) The vector \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) can be expressed as \( c_1y_1 + c_2y_2 + c_3y_3 \). Find \( c_1 \), \( c_2 \) and \( c_3 \).
Using \( c_i = \langle y_i | (1, 0, 0)^T \rangle / \langle y_i | y_i \rangle \) we get \( c_1 = 1/14 \), \( c_2 = 3/19 \), and \( c_3 = 11/284 \).

3. **When least is best.** Find all least-squares solutions to the system of equations

\[
\begin{align*}
x_1 + 2x_2 &= 1 \\
2x_1 + 4x_2 &= 1 \\
3x_1 + 6x_2 &= 1 \\
4x_1 + 8x_2 &= 1
\end{align*}
\]

Since \( A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{pmatrix} \) and \( b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \), we compute \( A^TA = \begin{pmatrix} 30 & 60 \\ 60 & 120 \end{pmatrix} \) and \( A^Tb = \begin{pmatrix} 10 \\ 20 \end{pmatrix} \). Note that \( A^TA \) is singular. Solving \( A^TAx = A^Tb \) gives infinitely many solutions: \( x = \left( \frac{1}{3}, 0 \right) + t \left( -2, 1 \right) \), where \( t \) is arbitrary.

4. **Working 24/7.** Consider the Hermitian matrix \( H = \begin{pmatrix} 24 & 7 \\ 7 & -24 \end{pmatrix} \).

a) Find the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) and corresponding eigenvectors \( b_1 \) and \( b_2 \) of \( H \).

The trace is zero and the determinant is \(-(25)^2\), so the eigenvalues are 25 and \(-25\), with eigenvectors \((7, 1)^T\) and \((1, -7)^T\).

b) Decompose \( x_0 = \begin{pmatrix} 13 \\ 9 \end{pmatrix} \) as a linear combination of \( b_1 \) and \( b_2 \).

The eigenvectors are orthogonal, so the coefficients are \( \langle b_1 | x_0 \rangle / \langle b_1 | b_1 \rangle = \frac{100}{50} = 2 \) and \( \langle b_2 | x_0 \rangle / \langle b_2 | b_2 \rangle = \frac{-50}{50} = -1 \), so \( x_0 = 2b_1 - b_2 \).

c) If \( dx/dt = Hx \) and \( x(0) = x_0 \) as in (b), what is \( x(t) \)?

\[ x(t) = 2e^{25t}b_1 - e^{-25t}b_2. \]