1) In $\mathbb{R}^2$, let $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard basis and let $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 7 \end{pmatrix} \right\}$ be an alternate basis.

a) Find $P_{\mathcal{EB}}$ and $P_{\mathcal{BE}}$.

\[ P_{\mathcal{EB}} = \left( \begin{array}{cc} 2 & 5 \\ 3 & 7 \end{array} \right), \quad P_{\mathcal{BE}} = P_{\mathcal{EB}}^{-1} = \left( \begin{array}{cc} -7 & 5 \\ 3 & -2 \end{array} \right). \]

Here we used the fact that \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)^{-1} = \frac{1}{ad-bc} \left( \begin{array}{cc} d & -b \\ -c & a \end{array} \right) \).

b) If $v = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, find $[v]_\mathcal{B}$.

\[ [v]_\mathcal{E} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad [v]_\mathcal{B} = P_{\mathcal{BE}} [v]_\mathcal{E} = \begin{pmatrix} -23 \\ 10 \end{pmatrix}. \]

c) Solve the system of equations: $2x_1 + 5x_2 = 4; \quad 3x_1 + 7x_2 = 1$.

This is the exact same problem as (b), namely writing $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$. The solution, as before, is $x_1 = -23, x_2 = 10$.

2. In $\mathbb{R}_1[t]$, let $\mathcal{E} = \{1, t\}$ be the standard basis and let $\mathcal{B} = \{4 + 5t, 3 + 4t\}$ be an alternate basis. Let $L : \mathbb{R}_1[t] \rightarrow \mathbb{R}_1[t]$ be the linear transformation $L(a_0 + a_1 t) = (16a_0 - 12a_1) + (20a_0 - 15a_1)t$.

a) Find $P_{\mathcal{EB}}$ and $P_{\mathcal{BE}}$.

\[ P_{\mathcal{EB}} = \left( \begin{array}{cc} 4 \\ 5 \\ 3 \\ 4 \end{array} \right), \quad P_{\mathcal{BE}} = P_{\mathcal{EB}}^{-1} = \left( \begin{array}{cc} 4 & -3 \\ -5 & 4 \end{array} \right). \]

b) Find the matrix of $L$ relative to the standard basis (that is, find $[L]_\mathcal{E}$).

\[ [L]_\mathcal{E} = ([L(e_1)]_\mathcal{E}, [L(e_2)]_\mathcal{E}) = \begin{pmatrix} 16 & -12 \\ 20 & -15 \end{pmatrix}. \]

c) Find the matrix of $L$ relative to the basis $\mathcal{B}$ (that is, find $[L]_\mathcal{B}$). [The answer to (c) is much simpler than the answer to (b) and illustrates why we use bases like $\mathcal{B}$.

There are two ways to do this. We could compute $L(b_1) = 4 + 5t = b_1$, hence $[L(b_1)]_\mathcal{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and compute $L(b_2) = 0$ to conclude that $[L]_\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, or we could compute $P_{\mathcal{BE}} [L]_\mathcal{E} P_{\mathcal{EB}}$ to get the same result.
3. Let \( A = \begin{pmatrix} 1 & 2 & 5 & 1 & 15 \\ 2 & 0 & 6 & 1 & 10 \\ 3 & 2 & 11 & 1 & 13 \\ 6 & 4 & 22 & 3 & 38 \end{pmatrix} \). \( A \) is row-equivalent to \( A_{\text{rref}} = \begin{pmatrix} 1 & 0 & 3 & 0 & -1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \).

a) Find a basis for the space of solutions to \( A \mathbf{x} = 0 \).

Our equations are \( x_1 = -3x_3 + x_5, \ x_2 = -x_3 - 2x_5, \ x_4 = -12x_5, \) and of course \( x_3 = x_3 \) and \( x_5 = x_5 \). In other words \( \mathbf{x} = x_3 \mathbf{b}_1 + x_5 \mathbf{b}_2, \) where

\[
\mathbf{b}_1 = \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ -12 \\ 1 \end{pmatrix}
\]

form a basis for our space of solutions.

b) Find a basis for the column space of \( A \).

These are the first, second, and fourth columns of \( A \), corresponding to the pivot columns of \( A_{\text{rref}} \). That is, \( \begin{pmatrix} 1 \\ 2 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \\ 4 \end{pmatrix}, \) and \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \). There are other possible bases, of course, but this is the one described in class and in Appendix A.

c) Let \( L : \mathbb{R}_4[t] \to \mathbb{R}_3[t] \) be a linear transformation with \( [L]_{\mathcal{E}} = A \). Here \( \mathcal{E} = \{1, t, t^2, t^3, t^4\} \) is the standard basis for \( \mathbb{R}_4[t] \) and \( \tilde{\mathcal{E}} = \{1, t, t^2, t^3\} \) is the standard basis for \( \mathbb{R}_3[t] \). Find bases for \( \text{Ker}(L) \) (the kernel of \( L \)) and \( \text{Range}(L) \).

This is essentially the same as parts (a) and (b)! A basis for the kernel of \( L \) is given by the vectors whose coordinates are the answer to (a), namely \( \{-3 - t + t^2, 1 - 2t - 12t^3 + t^4\} \), and a basis for the range are the vectors whose coordinates are the answer to (b), namely \( \{1 + 2t + 3t^2 + 6t^3, 2 + 2t^2 + 4t^3, 1 + t + t^2 + 3t^3\} \).

4. True or False? Each question is worth 4 points. You do NOT need to justify your answers, and partial credit will NOT be given.

a) If \( A \) is a matrix, then the pivot columns of \( A_{\text{rref}} \) form a basis for the column space of \( A \).

False. A basis is formed from the columns of \( A \), not the columns of \( A_{\text{rref}} \).

b) The set of solutions to \( A \mathbf{x} = 0 \) is the same as the set of solutions to \( A_{\text{rref}} \mathbf{x} = 0 \).
c) If $L : V \to V$ is an operator and $\mathcal{B}$ and $\mathcal{D}$ are bases for $V$, then $[L]_{\mathcal{B}} = P_{\mathcal{D}} [L]_{\mathcal{D}} P_{\mathcal{B}}$.

*False. You need to switch $P_{\mathcal{D}}$ and $P_{\mathcal{B}}$ to get the right formula.*

d) If $V$ is an $n$-dimensional vector space with a basis $\mathcal{B}$, then a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $V$ is linearly independent if and only if the matrix $A = ([\mathbf{v}_1]_{\mathcal{B}} \ldots [\mathbf{v}_k]_{\mathcal{B}})$ has rank $k$.

*True. You need a pivot in each of the $k$ columns.*

e) If $V$ is an $n$-dimensional vector space with a basis $\mathcal{B}$, then a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ spans $V$ if and only if the matrix $A = ([\mathbf{v}_1]_{\mathcal{B}} \ldots [\mathbf{v}_k]_{\mathcal{B}})$ has rank $n$.

*True. You need a pivot in each of the $n$ rows.*