1) Gram Schmidt:

a)(10 points) On \( \mathbb{R}^3 \) with the usual inner product, Use Gram-Schmidt to convert \( x_1 = (1, 2, 0)^T \), \( x_2 = (3, 1, 1)^T \), \( x_3 = (4, 3, -5)^T \) to an orthogonal basis.

\[
y_1 = x_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}
\]

\[
y_2 = x_2 - \frac{\langle y_1 | x_2 \rangle}{\langle y_1 | y_1 \rangle} y_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \frac{5}{5} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}
\]

\[
y_3 = x_3 - \frac{\langle y_1 | x_3 \rangle}{\langle y_1 | y_1 \rangle} y_1 - \frac{\langle y_2 | x_3 \rangle}{\langle y_2 | y_2 \rangle} y_2 = \begin{pmatrix} 4 \\ 3 \\ -5 \end{pmatrix} - \frac{10}{5} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - 0 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix}
\]

b)(15 points) On \( \mathbb{R}_2[t] \) with the inner product \( \langle f | g \rangle = \int_0^2 f(t) g(t) \, dt \), transform \( \{1, t, t^2\} \) to an orthogonal basis.

\[
y_1 = x_1 = 1
\]

\[
y_2 = x_2 - \frac{\int_0^2 t \, dt}{\int_0^2 1 \, dt} y_1 = t - 1
\]

\[
y_3 = x_3 - \frac{\int_0^2 t^2 \, dt}{\int_0^2 1 \, dt} y_1 - \frac{\int_0^2 t^2(t-1) \, dt}{\int_0^2 (t-1)^2 \, dt} y_2 = t^2 - \frac{4}{3}(t-1) = t^2 - 2t + \frac{2}{3}
\]

2. a)(15 points) Find the equation of the best line through the points \((1, -4), (2, 1), \) and \((3, 2)\).

\[
A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}, \quad A^T A = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}, \quad A^T b = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = (A^T A)^{-1}(A^T b) = \begin{pmatrix} -19/3 \\ 3 \end{pmatrix}, \quad \text{so the best line is} \quad y = 3x - 19/3.
\]

b)(10 points) Let \( V \) be the subspace of \( \mathbb{R}^3 \) that is the span of the vectors \( (1, 2, 3)^T \) and \( (1, 1, 1)^T \). Find the point in \( V \) that is closest to \( (-4, 1, 2)^T \).
This is essentially the same problem, since the least-squares solution to \(Ax = b\) places \(Ax\) as close as possible to \(b\). Our answer is \(A \left( \begin{array}{c} -19/3 \\ -1/3 \\ 8/3 \end{array} \right) \).

3. On \(\mathbb{C}^3\) with the usual inner product, let

\[
L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + ix_2 - ix_3 \\ 2x_2 + (1 - i)x_3 \\ ix_1 + 3x_2 + x_3 \end{pmatrix}
\]

a) (5 points) Find the matrix of \(L\):

\[
L = \begin{pmatrix} 1 & i & -i \\ 0 & 2 & 1 - i \\ i & 3 & 1 \end{pmatrix}
\]

b) (10 points) Let \(x = \begin{pmatrix} 1 \\ 10 \\ 100 \end{pmatrix}\). Compute \(L^\dagger(x)\). Since \(L^\dagger = \begin{pmatrix} 1 & 0 & -i \\ -i & 2 & 3 \\ 1 & i + 1 & 1 \end{pmatrix}\),

\[
L^\dagger x = \begin{pmatrix} 1 - 100i \\ 320 - i \\ 110 + 11i \end{pmatrix}.
\]

c) (10 points) Let \(V\) be the space of real-valued functions on the real line, with the inner product \(\langle f \mid g \rangle = \int_{-\infty}^{\infty} f(t)g(t) dt\). Let \(A : V \to V\) be the linear transformation \(A = t + d/dt\) (That is, \((A(f))(t) = tf(t) + f'(t))\). Let \(g(t) = e^{-t^2/2}\). Compute \(Ag\) and \(A^\dagger g\).

We saw in class that the adjoint to \(d/dt\) is \(-d/dt\), while multiplication by \(t\) is self-adjoint, so \(A^\dagger = t - d/dt\). It’s then an easy calculation to get \(Ag = 0\), \(A^\dagger g(t) = 2te^{-t^2/2}\). [Physics note: In quantum mechanics, \(g\) is the wave function of the ground state of a harmonic oscillator. The operators \(A\) and \(A^\dagger\) are called “ladder operators”, or “raising and lowering operators”. \(A^\dagger\) increases the energy level by one, and \(2te^{-t^2/2}\) is the wave function for the first excited state. \(A\) lowers the energy by one. Since there’s nothing below the ground state, we have \(Ag = 0\].]

4. Grab bag. These are short-answer or true/false questions. Each question is worth 5 points. You do NOT need to justify your answers, and partial credit will NOT be given.
a) True or false? The matrix \( \begin{pmatrix} 5 & 4i \\ -4i & -1 \end{pmatrix} \) has orthogonal eigenvectors.

True. The matrix is Hermitian.

b) True or false? The matrix \( \frac{1}{\sqrt{7}} \begin{pmatrix} 2 - i & -1 + i \\ 1 + i & 2 + i \end{pmatrix} \) is unitary.

True. The columns are orthonormal.

c) Let \( \mathbf{x}(t) \) be the solution to \( \frac{\text{d}\mathbf{x}}{\text{d}t} = A\mathbf{x} \), where \( A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & -1 & 4 \\ -2 & 1 & 0 & 5 \\ -3 & -4 & -5 & 0 \end{pmatrix} \) and \( \mathbf{x}(0) = (5, -3, 1, 1)^T \). Find the limit, as \( t \to \infty \), of \( |\mathbf{x}(t)| \). (This has a quick and easy solution, and you do NOT have to diagonalize \( A \)!

Since \( A \) is anti-symmetric, \( e^{At} \) is orthogonal, so \( \mathbf{x}(t) = e^{At}\mathbf{x}(0) \) has the same length as \( \mathbf{x} \), namely 6.

d) True or false? If a matrix \( M \) satisfies \( M = M^T \), then the eigenvalues of \( M \) are real.

False. Some of the matrix elements of \( M \) may be complex, in which case \( M \) won’t be Hermitian. (E.g., \( M \) could be \( i \) times the identity)

e) True or false? If a matrix is unitary, then it is not Hermitian.

False. The identity matrix is both Hermitian and unitary. More generally, any diagonalizable matrix with orthogonal eigenvectors and who eigenvalues are 1 and -1 is both Hermitian and unitary.