1) Find the eigenvalues of the matrix

\[
A = \begin{pmatrix}
5 & 3 & 0 & 0 \\
-3 & -1 & 0 & 3 \\
0 & 0 & 3 & 1 \\
0 & 0 & 4 & 3
\end{pmatrix}.
\]

For each eigenvalue, indicate what the geometric and algebraic multiplicity is. If the matrix is diagonalizable, find a basis of \( \mathbb{R}^4 \) consisting of eigenvectors of \( A \). If the matrix is not diagonalizable, do the next best thing and find a basis consisting of power vectors.

2. Do ONE and ONLY one of the following two problems. [If you attempt both, cross out the one you don’t want graded. If it’s not clear which you intend, I’ll grade (a) and ignore (b).]

a) Compute \( e^{At} \), where \( A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \).

b) Or compute \( B^n \), where \( B = \begin{pmatrix} 3 & 0 \\ 5 & 3 \end{pmatrix} \).

3. The matrix \( A = \begin{pmatrix} -2 & 3 \\ 2 & -1 \end{pmatrix} \) has eigenvalues \( \lambda_1 = 1 \) and \( \lambda_2 = -4 \) and corresponding eigenvectors \( b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( b_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \).

a) Solve the system of difference equations

\[
\begin{align*}
x_1(n) &= -2x_1(n-1) + 3x_2(n-1) \\
x_2(n) &= 2x_1(n-1) - x_2(n-1)
\end{align*}
\]

with initial conditions \( x(0) = \begin{pmatrix} -7 \\ 8 \end{pmatrix} \).

b) Solve the system of differential equations

\[
\begin{align*}
\frac{dx_1}{dt} &= -2x_1 + 3x_2 \\
\frac{dx_2}{dt} &= 2x_1 - x_2
\end{align*}
\]

with initial conditions \( x(0) = \begin{pmatrix} -7 \\ 8 \end{pmatrix} \).
c) Solve the system of second order differential equations

\[ \frac{d^2 x_1}{dt^2} = -2x_1 + 3x_2 \]
\[ \frac{d^2 x_2}{dt^2} = 2x_1 - x_2 \]

with initial conditions \( x(0) = \begin{pmatrix} -7 \\ 8 \end{pmatrix} \) and \( \dot{x}(0) = \begin{pmatrix} -5 \\ 5 \end{pmatrix} \).

4. A matrix \( A \) has eigenvalues \(-4\) and \(1/2\) and eigenvectors \( b_1 = \begin{pmatrix} 3 \\ 7 \end{pmatrix} \) and \( b_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \). Suppose that \( x(0) = c_1 b_1 + c_2 b_2 \), where \( c_1 \) and \( c_2 \) are both nonzero.

a) Describe the long-time behavior of the solution to \( x(n) = A x(n - 1) \). Which mode(s) grow(s) and which shrink? In what direction will \( x(n) \) be pointing in the long run, and how fast will it be growing or shrinking?

b) Describe the long-time behavior of the solution to \( \frac{dx}{dt} = A x \). Which mode(s) grow(s) and which shrink? In what direction will \( x(t) \) be pointing in the long run, and how fast will it be growing or shrinking?