1) (40 pts, 2 pages) Consider the point \( P(1, 4, -1) \), \( Q(2, 4, 3) \) and \( R(1, 5, -4) \) in \( \mathbb{R}^3 \).

a) Find the equation of the line \( L_1 \) through \( P \) and \( Q \). Express your answer both in vector form and in symmetric form.

Since \( \vec{PQ} = \langle 1, 0, 4 \rangle \), the line \( L_1 \), in vector form, is \( \mathbf{r}(t) = \langle 1, 4, -1 \rangle + t\langle 1, 0, 4 \rangle \). In symmetric form, it is \( y = 4 \) and \( \frac{x-1}{1} = \frac{z+1}{4} \), or equivalently \( z = 4x - 5 \).

b) Find the equation of the line \( L_2 \) through \( P \) and \( R \). Express your answer both in vector form and in symmetric form.

\[ \mathbf{r}(t) = \langle 1, 4, -1 \rangle + t\langle 0, 1, -3 \rangle \]

In symmetric form, it is \( x = 1 \) and \( \frac{y-4}{1} = \frac{z+1}{-3} \), or equivalently \( z = -3y + 11 \).

c) Find a vector perpendicular to both \( L_1 \) and \( L_2 \).

This is the cross product of \( \langle 1, 0, 4 \rangle \) and \( \langle 0, 1, -3 \rangle \), namely \( \mathbf{n} = \langle -4, 3, 1 \rangle \). (Any nonzero multiple of this vector is also a correct answer.)

d) Find the equation of a plane containing \( P \), \( Q \), and \( R \).

We can read off the equation of the plane from the form of \( \mathbf{n} \): \( 0 = -4(x-1) + 3(x-4) + z - (-1) = 0 \), or \(-4x + 3y + z = 7 \).

e) Find the equation of a line through \( P \) that is perpendicular to both \( L_1 \) and \( L_2 \).

In vector form, the line is \( \mathbf{r}(t) = \langle 1, 4, -1 \rangle + t\langle -4, 3, 1 \rangle \).

2. (30 points) Consider the parametrized curve \( \mathbf{r}(t) = \langle t - 4, t^2 + 5, \frac{2t^3 + 1}{3} \rangle \), where we think of \( t \) as time.

a) Find the velocity and speed as a function of \( t \).

The velocity is the derivative of \( \mathbf{r}(t) \), namely \( \langle 1, 2t, 2t^2 \rangle \). The speed is \( \sqrt{1 + 4t^2 + 4t^4} = 1 + 2t^2 \).

b) Find a unit vector tangent to the curve at the point \( \langle -3, 3, 1 \rangle \).

This is at \( t = 1 \). \( \mathbf{T} = \frac{\mathbf{r}'(1)}{||\mathbf{r}'(1)||} = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle \). Another possible answer is \(-\mathbf{T}\).

c) Find the arc-length of the portion of the curve that runs from \( t = 1 \) to \( t = 3 \).

\[ \text{Arc length} = \int_1^3 (1 + 2t^2) dt = 19 \frac{1}{3} = \frac{58}{3} \].

3. (30 pts, 2 pages)

Consider the function \( f(x, y) = e^{x-1} [1 + \ln(x + 3xy)] \).
a) Find the partial derivatives $f_x$, $f_y$, $f_{xx}$, $f_{xy}$ and $f_{yy}$ as functions of $x$ and $y$.

The derivatives are slightly easier if you rewrite the function as $f(x, y) = e^{x-1} [1 + \ln(x) + \ln(1 + 3y)]$, but of course the answers are the same whether you do this or not:

By the product rule, $f_x(x, y) = e^{x-1} [1 + \ln(x + 3xy)] + \frac{e^{x-1}}{x}$.

$f_y(x, y) = \frac{3e^{x-1}}{1 + 3y}$.

$f_{xx}(x, y) = e^{x-1} [1 + \ln(x + 3xy)] + 2e^{x-1} - \frac{e^{x-1}}{x^2}$.

$f_{xy}(x, y) = \frac{3e^{x-1}}{1 + 3y}$. (This is most easily computed as the derivative of $f_y$ with respect to $x$.)

$f_{yy}(x, y) = \frac{-9e^{x-1}}{(1 + 3y)^2}$.

b) Find the equation of the plane tangent to the surface $z = f(x, y)$ at $(1, 0, 1)$.

Since $f(1, 0) = 1$ and $f_x(1, 0) = 2$ and $f_y(1, 0) = 3$, our tangent plane is $z = 1 + 2(x - 1) + 3y = 2x + 3y - 1$.

c) Use first derivatives to estimate the value of $f(0.99, 0.02)$.

Plugging in $x = 0.99$ and $y = 0.02$ into the equation of the tangent plane gives $z = 1 + 2(-0.01) + 3(0.02) = 1.04$, so $f(0.99, 1.02) \approx 1.04$.

Extra credit (5 points): Use second derivatives to get a better estimate of $f(0.99, 0.02)$. (Yes, I expect you to do the arithmetic by hand)

We use the quadratic approximation:

\[
\begin{align*}
    f(0.99, 1.02) & \approx f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y) \\
                   & \quad + \frac{1}{2} f_{xx}(1, 0)(x - 1)^2 + f_{xy}(1, 0)(x - 1)y + \frac{1}{2} f_{yy}(1, 0)y^2 \\
                   & = 1 + 2(-0.01) + 3(0.02) + 1(-0.01)^2 + 3(-0.01)(0.02) - \frac{9}{2}(0.02)^2 \\
                   & = 1 - 0.02 + 0.06 + 0.0001 - 0.0018 - 0.0006 = 1.0377
\end{align*}
\]

This is accurate to one part in 10,000. From a calculator I get that $f(0.99, 0.02) \approx 1.03778862$. 
