M408M Second Midterm Exam, October 29, 2013

1) (40 pts, 2 pages) Consider the point $P(1,4,-1)$, $Q(2,4,3)$ and $R(1,5,-4)$ in $\mathbb{R}^3$.

a) Find the equation of the line L_1 through P and Q. Express your answer both in vector form and in symmetric form.

Since $\vec{PQ} = \langle 1,0,4\rangle$, the line L_1, in vector form, is $\mathbf{r}(t) = \langle 1,4,-1\rangle + t\langle 1,0,4\rangle$. In symmetric form, it is $y = 4$ and $\frac{x-1}{1} = \frac{z+1}{4}$, or equivalently $z = 4x - 5$.

b) Find the equation of the line L_2 through P and R. Express your answer both in vector form and in symmetric form. $\mathbf{r}(t) = \langle 1,4,-1\rangle + t\langle 0,1,-3\rangle$. In symmetric form, it is $x = 1$ and $\frac{y-4}{1} = \frac{z+1}{-3}$, or equivalently $z = -3y + 11$.

c) Find a vector perpendicular to both L_1 and L_2.

This is the cross product of $\langle 1,0,4\rangle$ and $\langle 0,1,-3\rangle$, namely $\mathbf{n} = \langle -4,3,1\rangle$. (Any nonzero multiple of this vector is also a correct answer.)

d) Find the equation of a plane containing P, Q, and R.

We can read off the equation of the plane from the form of $\mathbf{n}$: $0 = -4(x-1) + 3(x-4) + z - (-1) = 0$, or $-4x + 3y + z = 7$.

e) Find the equation of a line through P that is perpendicular to both L_1 and L_2. (You only have to give one form of the equation of this line — your pick which one.)

In vector form, the line is $\mathbf{r}(t) = \langle 1,4,-1\rangle + t\langle -4,3,1\rangle$.

2. (30 points) Consider the parametrized curve $\mathbf{r}(t) = \langle t-4, t^2+5, \frac{2t^3+1}{3}\rangle$, where we think of t as time.

a) Find the velocity and speed as a function of t.

The velocity is the derivative of $\mathbf{r}(t)$, namely $\langle 1, 2t, 2t^2\rangle$. The speed is $\sqrt{1+4t^2+4t^4} = 1 + 2t^2$.

b) Find a unit vector tangent to the curve at the point $\langle -3,6,1\rangle$.

This is at $t = 1$. $\mathbf{T} = \frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\rangle$. Another possible answer is $-\mathbf{T}$.

c) Find the arc-length of the portion of the curve that runs from $t = 1$ to $t = 3$.

Arc length $= \int_1^3 (1+2t^2)dt = 19\frac{1}{3} = \frac{58}{3}$.

3. (30 pts, 2 pages)

Consider the function $f(x,y) = e^{x-1}\left[1 + \ln(x+3xy)\right]$.

a) Find the partial derivatives f_x, f_y, f_{xx}, f_{xy} and f_{yy} as functions of x and y.

The derivatives are slightly easier if you rewrite the function as $f(x,y) = e^{x-1}\left[1+\ln(x)+\ln(1+3y)\right]$, but of course the answers are the same whether you do this or not:

By the product rule, $f_x(x,y) = e^{x-1}\left[1+\ln(x+3xy)\right] + \frac{e^{x-1}}{x}$.

$f_y(x,y) = \frac{3e^{x-1}}{1+3y}$.

$f_{xx}(x,y) = e^{x-1}\left[1+\ln(x+3xy)\right] + \frac{2e^{x-1}}{x} - \frac{e^{x-1}}{x^2}$.

$f_{xy}(x,y) = \frac{3e^{x-1}}{1+3y}$. (This is most easily computed as the derivative of f_y with respect to x.)

$f_{yy}(x,y) = \frac{-9e^{x-1}}{(1+3y)^2}$.

b) Find the equation of the plane tangent to the surface $z = f(x,y)$ at $(1,0,1)$.

Since $f(1,0) = 1$ and $f_x(1,0) = 2$ and $f_y(1,0) = 3$, our tangent plane is $z = 1 + 2(x-1) + 3y = 2x + 3y - 1$.

c) Use first derivatives to estimate the value of $f(0.99, 0.02)$.

Plugging in $x = 0.99$ and $y = 0.02$ into the equation of the tangent plane gives $z = 1 + 2(-0.01) + 3(0.02) = 1.04$, so $f(0.99, 1.02) \approx 1.04$.

Extra credit (5 points): Use second derivatives to get a better estimate of $f(0.99, 0.02)$. (Yes, I expect you to do the arithmetic by hand)

We use the quadratic approximation:

$$\begin{aligned}
f(0.99, 1.02) &\approx f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y) \\
&\quad + \frac{1}{2}f_{xx}(1,0)(x-1)^2 + f_{xy}(1,0)(x-1)y + \frac{1}{2}f_{yy}(1,0)y^2 \\
&= 1 + 2(-0.01) + 3(0.02) + 1(-0.01)^2 + 3(-0.01)(0.02) - \frac{9}{2}(.02)^2 \\
&= 1 - 0.02 + 0.06 + 0.0001 - 0.0018 - 0.0006 = 1.0377
\end{aligned}$$

This is accurate to one part in 10,000. From a calculator I get that $f(0.99, 0.02) \approx 1.03778862$.