Lie Groups, Problem Set # 2 Solutions

This week's problems were all from the book, namely Section 2.1, problems 5, 6 and 9, and 2.2, problems 4, 5 and 7.

Problem 2.1.5: (a) First note that $ij = k = -ji$, so for any complex number α, $j\alpha = \bar{\alpha}j$ and $\alpha j = j\bar{\alpha}$. If $q_1 = \alpha + j\beta$ and $q_2 = \gamma + j\delta$, then $\bar{q}_1 = \bar{\alpha} + \bar{\beta}\bar{j} = \bar{\alpha} - j\beta$, whose matrix is the adjoint of the matrix of q_1. Likewise, $q_1 q_2 = \alpha\gamma + j\beta\gamma + \alpha j\delta + j\beta j\delta = (\alpha\gamma - \bar{\beta}\delta) + j(\beta\gamma + \bar{\alpha}\delta)$, whose matrix is $\begin{pmatrix} \alpha\gamma - \bar{\beta}\delta & -\bar{\beta}\bar{\gamma} - \alpha\bar{\delta} \\ \beta\gamma + \bar{\alpha}\delta & \bar{\alpha}\bar{\gamma} - \beta\bar{\delta} \end{pmatrix}$, which is the product of the matrix of q_1 and the matrix of q_2. Since quaternionic multiplication is mapped to multiplication of complex matrices, this gives a homomorphism from $Sp(1)$ (aka the unit quaternions) to the matrices of the given form with $|\alpha|^2 + |\beta|^2 = 1$, and the homomorphism is clearly 1–1. But by example 2 the image is precisely $SU(2)$.

(b) If $\bar{\gamma} = -\gamma$, then the matrix of γ (call it M_γ) is anti-Hermitian, so it has pure imaginary eigenvalues and orthogonal eigenvectors. By choosing the phases of the two eigenvectors correctly, we can write $M_\gamma = \lambda P D P^{-1}$, where $P \in SU(2)$, λ is real and positive and $D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Likewise, we can write $M_j = P_0 D P_0^{-1}$, so $D = P_0^{-1} M_j P_0$. We then have $M_\gamma = \lambda P P_0^{-1} M_j P_0 P^{-1}$. If we take α to be the quaternion whose matrix is $\sqrt{\lambda} P_0 P^{-1}$, then $M_\gamma = M_{\bar{\alpha}} M_j M_\alpha$, so $\gamma = \bar{\alpha} j \alpha$.

Note that α is not uniquely defined. Replacing $\alpha' = e^{j\phi}\alpha$ would work as well. This ambiguity corresponds the the phase freedom we have in choosing the eigenvectors of M_γ.

Problem 2.1.6: (a) $\mathbf{n} = \mathbf{h}(\mathbf{3}, \mathbb{R})$ is just the upper triangular matrices, since if X is not upper-triangular, then $\exp(tX) \approx 1 + tX$ is not in the group for small t. Notice that the three basis vectors for $\mathbf{n}$ (call them e_α, e_β and e_γ) have all pairwise products equal to zero, except $e_\alpha e_\beta$, which equals e_γ. In particular, the bracket of any two matrices is a multiple of $e_\gamma \subset h(3, \mathbb{R})$. (b) If $X \in \mathbf{n}$, then $X^3 = 0$, and $\exp(X) = 1 + X + X^2/2$ is in $H(3, R)$. Likewise, if $a \in H(3, R)$, then $(a-1)^3 = 0$, so $\log(a) = a - 1 - (a-1)^2/2$, which is easily seen to be in $h(3, R)$. (c) The brackets in $\mathbf{n}$ are: $[e_\alpha, e_\beta] = e_\gamma$, $[e_\alpha, e_\gamma] = [e_\beta, e_\gamma] = 0$. Note that $[X, [Y, Z]] = 0$ for any $X, Y, Z \in \mathbf{n}$. If $Y = \begin{pmatrix} 0 & y_1 & y_3 \\ 0 & 0 & y_2 \\ 0 & 0 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} 0 & x_1 & x_3 \\ 0 & 0 & x_2 \\ 0 & 0 & 0 \end{pmatrix}$, then $Ad(\exp(X))Y = Y + [X, Y] = Y + (y_1 x_2 - y_2 x_1) e_\gamma$, since all higher-order brackets are zero. The adjoint orbit of Y is therefore: (i) Y itself, if $y_1 = y_2 = 0$. In this case Y is proportional to e_γ, and commutes with all elements of the group. (ii) Y plus an arbitrary multiple of e_γ, if

$y_1 \neq 0$ or $y_2 \neq 0$.

Problem 2.1.9: (a) In the Gram-Schmidt process we construct an orthogonal basis $\{w_1, w_2, \ldots, w_n\}$ from an arbitrary basis $\{v_1, \ldots, v_n\}$. in such a way that each w_k equals v_k minus a linear combination of the previous w_j's. Turning things around, each v_k equals w_k plus a linear combination of the previous w_j's. Let W be a matrix whose columns are the w's, and V be a matrix whose columns are the v's. Then we have $V = W\tilde{b}$, where $\tilde{b}$ is an upper triangular matrix with 1's on the diagonal. We can further write $W = Ud$, where d is diagonal, with positive entries, and the columns of U are orthonormal. Setting $b = d\tilde{b}$, we have $V = Ub$, with $U \in O(n)$ and $b \in B$.

Note that the Gram-Schmidt process is deterministic. Each basis $\{v_j\}$ is associated with exactly one pair (U, b), and of course each pair (U, b) is associated with one basis – the columns of $V = Ub$. This shows that every invertible matrix can be uniquely written as the product of an orthogonal matrix and an upper-triangular matrix with positive diagonal entries.

(b) Since b has positive determinant, each element of $GL(n, \mathbb{R})_+$ is associated with a unique pair (U, b) with $U \in SO(n)$ and $b \in B$. Note that B is convex, and hence contractible (and connected). Likewise, $SO(n)$ is connected. Given $V_0 = U_0 b_0$ and $V_1 = U_1 b_1$, just pick a path U_t from U_0 to U_1 and a path b_t from b_0 to b_1 and set $V_t = U_t b_t$. As for analyticity, we know that $\exp : so(n) \to SO(n)$ is onto. Pick elements X_0 and X_1 in $so(n)$ such that $\exp(X_0) = U_0$ and $\exp(X_1) = U_1$, and let $U_t = \exp(tX_0 + (1-t)X_1)$. Then pick $b_t = t(b_1) + (1-t)b_0$.

Problem 2.2.4: (a) The only sub-algebras of $so(3)$ are either 1-dimensional (with a trivial bracket), or the full 3-dimensional algebra. To see this, recall that the bracket in $so(3)$ is essentially the same thing as the cross product in $\mathbb{R}^3$. If X and Y are linearly independent, then $[X, Y]$ corresponds to a vector orthogonal to both $\vec{X}$ and $\vec{Y}$, and hence linearly independent of $\{\vec{X}, \vec{Y}\}$. Thus if any algebra has dimension greater than 1, it must have dimension 3.

(b) Even though $sl(2, C)$ is the complexification of $so(3)$, the set of available Lie sub-algebras is actually MORE than the complexification of the answer to (a). There exist 2-dimensional subalgebras, all of which are conjugate to the span of H and X_+. To see that these are the ONLY 2-dimensional subalgebras, we argue as follows:

Suppose we have a basis for a 2-D subalgebra, spanned by matrices A and B. Then $[A, B]$ is a linear combination of A and B. By calling this combination our second basis vector and rescaling our vectors, we can assume that $[A, B] = 2B$. If B is semi-simple and has eigenvalues $\pm\lambda$, then $\exp(2\pi B/\lambda) = 1$, so $Ad(\exp(2\pi B/\lambda))A = A$. But by Baker-Campbell-Haussdorff, $Ad(\exp(Bt)A = A + 2Bt$. So B must not be semi-simple, which implies it must be nilpotent, hence conjugate to X_+. The equation $[A, B] = 2B$ then implies that $A = H$ plus a multiple of B, so our algebra is spanned by H and X_+.

Problem 2.2.5. First consider the 1-dimensional sub-algebras. This is basically classifying 2×2 traceless non-zero real matrices up to scaling and conjugation. There are three classes, up to conjugacy by $SL(2, R)$: (i) Those with real eigenvalues (and real eigenvectors), conjugate to (a multiple of) H, (ii) Those with imaginary eigenvalues, conjugate (with a real change-of-basis) to a multiple of $X_- - X_+$, and the non-diagonalizable elements, conjugate to X_+. Next, the 2-dimensional sub-algebras. As with $sl(2, \mathbb{C})$, we have the span of A and B, with $[A, B] = 2B$ and $B = X_+$ (up to conjugacy). But then $A = H$ plus a multiple of X_+, so we have the span of H, X_+. In other words, all 2-dimensional sub-algebras are conjugate to the upper triangulars. (b) As abstract algebras, all 1-dimensional sub-algebras are isomorphic, and there is only one 2-dimensional algebra (up to conjugation), so all 2D sub-algebras are isomorphic.

Problem 2.2.7. If $a(t)$ is a family of automorphisms, then $a(x \cdot y) = (ax) \cdot (ay)$. Taking a derivative with respect to t at $t = 0$ and a =identity, we get $D(x \cdot y) = (Dx) \cdot y + x \cdot (Dy)$, where $D = a'(0)$. Thus D is a derivation. Conversely, suppose that D is a derivation. Let $a(t) = \exp(Dt)$, so $a' = aD$. Then $d/dt[(a(x \cdot y) - (ax) \cdot (ay)] = aD(x \cdot y) - (aDx) \cdot (ay) - (ax) \cdot (aDy) = a(Dx \cdot y + x \cdot Dy) - a(Dx \cdot y) - a(x \cdot Dy) = 0$, so $a(x \cdot y) - (ax) \cdot (ay)$ is constant. Since it is zero at $t = 0$, it is zero for all t.