Lie Groups, Problem Set # 4 Solutions

Like last week, this week's problems were all from the book, namely Section 2.6, problems 3, 5, 6 and 7, Section 2.7 problem 5 and Section 3.1 problem 1.

Section 2.6

2.6.3: There's a *really* cheap solution and a not-quite-so-cheap solution. The really cheap solution is to say that a bunch of maps $\phi_k : \mathbf{g} \to \mathbf{h}_k$ is the same as a single map $\mathbf{g} \to \oplus \mathbf{h}_k$, and that $\oplus_k \mathbf{h}_k$ is the Lie algebra of the linear group $H = H_1 \times H_2 \times \cdots \times H_k$. The result then follows from theorem 9. (Note that this only works for a finite collection of maps, since an infinite product of linear groups is not a linear group.)

The not-so-cheap solution is to mimick the proof of theorem 9 with multiple groups H_k. Let $\mathbf{g}' = \{(X, \phi_1(X), \ldots, \phi_n(X))\} \subset \mathbf{g} \oplus \mathbf{h}_1 \oplus \cdots \oplus \mathbf{h}_n$. Let $G' = \Gamma(\mathbf{g}')$. This is a subgroup of $G \times H_1 \times \cdots \times H_n$, and hence is a linear group. Let π_0 and f_k be the restrictions to G' of the projections from $G \times H_1 \times \cdots H_n$ to G and H_k, respectively. Since $L(\pi_0) : \mathbf{g}' \to \mathbf{g}$ is an isomorphism, π_0 is a local isomorphism, so G' is a cover of G. Meanwhile, $L(f_k)$ sends $(X, \phi_1(X), \ldots)$ to $\phi_k(X)$, so $L(f_k) = \phi_k \circ L(\pi_0)$.

2.6.5: The notation can get pretty ugly, so I'll write a solution for the specific case of homogeneous cubic polynomials. The generalization to other powers should be pretty clear.

(a) A general cubic polynomial is $f(x) = \sum_{ijk} P_{ijk} \xi_i \xi_j \xi_k$, with P_{ijk} a fixed set of coefficients. By the product rule, the derivative of this is $\sum_{ijk} P_{ijk}(\xi_i' \xi_j \xi_k + \xi_i \xi_j' \xi_k + \xi_i \xi_j \xi_k')$. Note that the coefficient of ξ_ℓ' is exactly $\partial f / \partial \xi_\ell$, taking into account the fact that i, j and k can independently equal ℓ. In other words, the derivative of f is $\sum_\ell \xi_\ell' \partial f / \partial \xi_\ell$.

Now consider a path $a(t) \in GL(n, \mathbb{R})$ through the identity whose derivative at $t = 0$ is X, and let $a^{-1} x = \sum \xi_j e_j$. Then $\xi_i' = -\sum_j X_{ij} \xi_j$, with the minus sign because we are looking at $a^{-1}x$, not ax. This makes the derivative of $f(a^{-1}x)$ equal to $\sum_{ij} -X_{ij} \xi_j \frac{\partial f}{\partial \xi_i}$, as required.

(b) Invariance means that $T(\exp(tX))$ acts trivially for every t and X, which is equivalent (by the usual differentiation-exponentiation game) to $\tau(X) f$ being zero for every X. Since τ is a linear function of X, we only need to check this for a basis of $so(3)$, namely the matrices $E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ and

$E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, corresponding to infinitesimal rotations about the x, y and z axes. But that's just saying $\xi_i \frac{\partial f}{\partial \xi_j} = \xi_j \frac{\partial f}{\partial \xi_i}$ for $(ij) = (23)$, (31) and (12). By the way, in quantum mechanics the operators $y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$, $z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$, $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$, are (up to factors of $\hbar$ and i) associated with angular momentum in the x, y and z directions.

There are several natural generalizations. One is to extend from polynomials to all analytic functions, or simply to all smooth functions. This isn't quite within the theory of linear groups, because in that case F is infinite-dimensional. Another generalization is to work with $SO(n)$ and say that a polynomial function of n variables is rotationally invariant if and only if $\xi_i \frac{\partial f}{\partial \xi_j} = \xi_j \frac{\partial f}{\partial \xi_i}$ for every pair (ij) of indices. In n dimensions there are $\binom{n}{2}$ components to the "angular momentum".

2.6.6. The double cover $SL(2,\mathbb{C}) \to SO(3,\mathbb{C})$ is just the complexification of the double cover $SU(2) \to SO(3)$. Specifically, $sl(2,\mathbb{C})$ is the space of traceless 2×2 matrices, which is the complexification of $su(2)$ and is isomorphic to $\mathbb{C}^3$. Ad gives a map from $SL(2,\mathbb{C})$ to $SL(sl(2,\mathbb{C})) = SL(3,\mathbb{C})$, but in fact the image is the much smaller group $SO(3,\mathbb{C})$.

To see this, look at the map $L(Ad) = ad$. If $X \in sl(2,\mathbb{C})$, we can write $X = Y + iZ$, where $Y, Z \in su(2)$. But then $ad(X) = ad(Y) + iad(Z)$. Since $ad(X)$ and $ad(Y) \in so(3)$, $ad(X) + iad(Y)$ is in the complexification of $so(3)$, namely $so(3,\mathbb{C})$. In fact, ad gives an isomorphism of $sl(2,\mathbb{C})$ and $so(3,\mathbb{C})$, implying that $Ad : SL(2,\mathbb{C}) to SO(3,\mathbb{C})$ is a covering map. But the kernel of this map is ± 1, so the map is a double cover.

2.6.7 (a) On $sl(2,\mathbb{R})$ there is an Ad-invariant inner product of signature $(2,1)$, namely $\langle X|Y\rangle = Tr(XY)/2$. The matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ form an orthonormal basis for this inner product, only with the third matrix having $\langle Z|Z\rangle = -1$ instead of $+1$. Since $sl(2,\mathbb{R})$ is a 3-dimensional real vector space with an inner product of signature (2,1), and since Ad maps $SL(2,\mathbb{R})$ to $SL(sl(2,\mathbb{R})) = SL(3)$ in a way that preserves that inner product, Ad maps $SL(2,\mathbb{R})$ to the identity component $SO_o(2,1)$ of $SO(2,1)$. By looking at the Lie map ad, we see that this map is locally bijective, hence a covering map. (It's not a covering map of $SO(2,1)$ since $SO(2,1)$ is disconnected, but it's a covering map of the identity component of $SO(2,1)$.) The kernel is ± 1 as usual, since ± 1 are the only matrices in $SL(2,\mathbb{R})$ that commute with all of $sl(2,\mathbb{R})$.

(b) This is similar. The Lie Algebra $su(1,1)$ is spanned by $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

and $\begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix}$. Just as with $sl(2,\mathbb{R})$, the Ad-invariant inner product $\langle X|Y\rangle = Tr(XY)/2$ has signature (2,1), since the first two basis elements square to +1 and the last one squares to -1. Thus Ad gives a map from $SU(1,1)$ so $SO(\mathbf{g}) = SO(2,1)$. The kernel is ± 1, and the map is onto (since both are 3-dimensional and connected).

(c) Let $a = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. You can explicitly check that $Ad(g)$ sends the three basis elements of $su(1,1)$ to the three basis elements of $sl(2,\mathbb{R})$, and so sends $SU(1,1)$ to $SL(2,\mathbb{R})$. Incidentally, this version of $SU(1,1)$ is *not* the automorphisms of the Hermitian form with matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Instead, it's $Aut(\phi)$ with $\tilde{\phi} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, which is conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

2.7.5: The "if" part is easy. If X is an imaginary diagonal matrix with λ_1/λ_2 irrational, then $\exp(tX)$ is a diagonal matrix with the first two eigenvalues $\exp(t\lambda_1)$ and $\exp(t\lambda_2)$. Such exponentials are bounded, so if the group is closed then it is compact, and the image of any continuous homomorphism must be compact. However, the projection onto the upper left 2×2 block has an image set that is not compact (or even closed), being a line that wraps irrationally around a 2-torus.

For the "only if" part, suppose that X is a matrix such that $\exp(tX)$ is not closed. This means that is an invertible matrix Y and a sequence t_i such that $\exp(t_i X)$ converges to Y. Since the eigenvalues of $\exp(tX)$ are exponentials of eigenvalues of X, this means that all of the eigenvalues of X must be pure imaginary, since otherwise $\exp(t\lambda)$ would either go to 0 or ∞, and could not converge to an eigenvalue of Y. Moreover, X must be diagonalizable, since the exponential of a nontrivial Jordan block grows without bound, and so cannot converge to a part of Y.

Once we have that X is (conjugate to) a pure imaginary diagonal matrix, we just have to consider the eigenvalues. If all eigenvalues are integer multiples of a number $i\lambda_0$, then $\exp(tX)$ is periodic with period $2\pi/\lambda_0$, and is isomorphic to a circle. Hence it is compact, and in particular closed. If two eigenvalues are not rationally related, then we have already shown that it is not closed.

3.1.1:(a) Given a real vector space E, let $F = E \oplus iE$, and let $C(v_R + iv_I) = v_R - iv_I$, where v_R and v_I are arbitrary elements of E. Then C is anti-linear and $C^2 = 1$.

Conversely, suppose we are given F and C. Since $C^2 = 1$, the eigenvalues of C are ± 1. Since C is anti-linear, multiplication by i sends the +1 eigenspace to the -1 eigenspace, and vice-versa. Let E be the +1 eigenspace. We then have $F = E \oplus iE$.

Thus the construction of E is the inverse of the operation described in the previous paragraph.

(b) If E is a (right) quaternionic vector space, let $F = E$ as a set. I will call the basic quaternions $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ to distinguish $\hat{\imath}$ from the complex number i. Define an action of the complex numbers on F by $(a + bi)v = v(a + b\hat{\imath})$, and define $J(v) = v\hat{\jmath}$. This makes F into a complex vector space. Since $(a+b\hat{\imath})\hat{\jmath} = \hat{\jmath}(a - b\hat{\imath})$, $J(\alpha v) = \bar{\alpha}Jv$, and since $\hat{\jmath}^2 = -1$, $J^2 = -1$.

Conversely, suppose we have a pair (F, J) with $J^2 = -1$ and J anti-linear. Define $v(a + b\hat{\imath} + c\hat{\jmath} + d\hat{k})$ to be $(a + bi)v + (c - di)J(v)$. This makes F into a quaternionic vector space.