Lie Groups, Problem Set # 8
Due Thursday, November 8

In this problem set we're going to study some homogeneous spaces.

1) Let V be an n-dimensional vector space over a field $K = \mathbb{R}$ or $\mathbb{C}$. The *Grassmanian* $Gr_{n,k}$ is the set of k-dimensional subspaces of V. $Gr_{n,1}$ is also called a *projective space*, $P^{n-1}(K)$ (often written as $\mathbb{C}P^{n-1}$ or $\mathbb{R}P^{n-1}$). Show that $Gr_{n,k}$ is an analytic manifold. What is its dimension? What are the coordinate charts?

Let A and B be $k \times n$ matrices of rank k. The row spaces of A and B are k-dimensional subspaces of $\mathbb{R}^n$ (or $\mathbb{C}^n$), and these row spaces are the same if and only if A and B are row-equivalent. $Gr_{n,k}$ is thus the space of rank k $k \times n$ matrices, mod row equivalence. (Normally I write vectors as columns rather than rows, but talking about the column space of A^T instead of the row space of A is too clumsy.)

Since the rank is k, we can find k linearly independent columns. Each choice of columns gives us a coordinate chart, described by matrices where these columns form the $k \times k$ identity matrix and the remaining columns are free. This is an affine space of (real or complex) dimension $k(n-k)$. For instance, $Gr_{4,2}$ consists of 6 charts, consisting of (equivalence classes of) matrices of the form

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}, \quad \begin{pmatrix} 1 & * & 0 & * \\ 0 & * & 1 & * \end{pmatrix}, \quad \begin{pmatrix} 1 & * & * & 0 \\ 0 & * & * & 1 \end{pmatrix},$$
$$\begin{pmatrix} * & 1 & 0 & * \\ * & 0 & 1 & * \end{pmatrix}, \quad \begin{pmatrix} * & 1 & * & 0 \\ * & 0 & * & 1 \end{pmatrix}, \quad \begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix},$$

where $*$ denotes an arbitrary number. The transition functions are analytic in the matrix entries. For instance,

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{a}{c} & 0 & b - \frac{ad}{c} \\ 0 & \frac{1}{c} & 1 & \frac{d}{c} \end{pmatrix}.$$

This makes $Gr_{n,k}$ into a (real or complex) $k(n-k)$-dimensional analytic manifold.

2) Show that $Gr_{n,k}$ is homeomorphic to $Gr_{n,n-k}$.

Pick an inner product on $\mathbb{R}^n$ or $\mathbb{C}^n$. Every k-plane has an orthogonal complement. Taking orthogonal complements is a continuous map from $Gr_{n,k}$ to $Gr_{n,n-k}$, and taking orthogonal complements again is the inverse map. The maps in both directions are real analytic, and in particular are continuous. However, with complex coefficients the map is *not* holomorphic, since it involves complex conjugation. (E.g., in $\mathbb{C}^2$ the orthogonal complement to the line spanned by $\begin{pmatrix} 1 \\ \alpha \end{pmatrix}$ is spanned by $\begin{pmatrix} \bar{\alpha} \\ -1 \end{pmatrix}$.)

3) Express $Gr_{n,k}$ as a quotient of $GL(V)$ by a subgroup H of $GL(V)$.

We need to find the little group for a particular point in $Gr_{n,k}$. Let V_0 be the k-plane spanned by the standard basis vectors $e_1, \ldots, e_k$. A matrix preserves V_0 if and only if it of the form $X = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where A is an invertible $k \times k$ matrix, C is an invertible $(n-k) \times (n-k)$ matrix, and B is an arbitrary $k \times (n-k)$ matrix. The space of such block-upper-triangular matrices is our group H, and $Gr_{n,k} = GL(n)/H$.

4) Express $Gr_{n,k}$ as a quotient of $U(V)$ (meaning $O(V)$ if V is real) by a different subgroup.

The action of $GL(n)$ described above is clearly transitive, but it is also overkill. We can move an arbitrary k-plane to another arbitrary k-plane with ordinary rotations. Then the little group is the set of rotations that leave both the subspace V_0 and its orthogonal complement fixed. This is $U(k) \times U(n-k)$. So $Gr_{n,k}(\mathbb{C}) = U(n)/(U(k) \times U(n-k))$ and $Gr_{n,k}(\mathbb{R}) = O(n)/(O(k) \times O(n-k))$.

5) If V is a real vector space, then the *oriented Grassmanian* $Gr^+_{n,k}$ is the space of *oriented* k-planes in V. Show that Gr^+ is a homogeneous space. What is $Gr^+_{n,1}$?

An oriented k-plane is described by an oriented basis. We still have $O(n)$ acting transitively, and in fact $SO(n)$ acts transitively, since we can simultaneously flip the sign of a basis element of V_0 and a basis element of the complement of V_0, while leaving everything else alone. If $G = SO(n)$, then $H = SO(k) \times SO(n-k)$, and we have $Gr^+_{n,k} = \frac{SO(n)}{SO(k) \times SO(n-k)}$. When $k = 1$, the oriented Grassmanian is just the sphere S^{n-1} (since a unit vector determines an oriented line and vice-versa), and we have $S^{n-1} = SO(n)/SO(n-1)$.

6) Let $d_1 < d_2 < \cdots < d_j < n$, where n is the dimension of V. (For simplicity, take $V = K^n$.) The *flag manifold* (or flag variety) $F(d_1, \ldots, d_j, K)$ of index $(d_1, \ldots, d_j = n)$ is the space of $(j+1)$-tuples of vector spaces $V_0, V_1, \ldots, V_j$, where V_0 is a trivial (0-dimensional) vector space, $V_j = V$, each V_i is a subspace of V_{i+1}, and each V_i has dimension d_i. (When $j = 2$, this is the same as a Grassmanian.) Show that $F(d_1, \ldots, d_j, \mathbb{R})$ is a homogeneous space G/H in two different ways: a) with $G = GL(V)$, and b) with G being a compact linear group.

a) $G = GL(n)$ and H is the space of block upper-triangular matrices where we partition $\{1, \ldots n\}$ into blocks of size d_1, $d_2 - d_1$, $d_3 - d_2$, etc.

b) $G = O(n)$ and $H = O(d_1) \times O(d_2 - d_1) \times O(d_3 - d_2) \times \times O(n - d_{j-1})$. For oriented flags we would have $G = SO(n)$ and $H = SO(d_1) \times \cdots \times SO(n - d_{j-1})$.

7) Repeat for complex flag manifolds. (This is almost identical.)

(a) is EXACTLY the same as for real flag manifolds, while (b) has $U(n)$ and $U(d_1)$, etc. instead of $O(n)$ and $O(d_1)$, etc.

8) The oriented flag is similar, only with oriented subspaces of V. Show that the oriented flag $F^+(1,2,3,\mathbb{R})$ is homeomorphic to $O(3)$.

The oriented flag is $SO(3)/(SO(1)^3)$. But $SO(1)$ is the trivial group!

More explicitly, a matrix in $SO(3)$ consists of three orthonormal columns. The first defines an oriented line. The first two define an oriented plane. The third is the cross product of the first two, and carries no new information.

The last example illustrates how flag manifolds got their name. A 1-dimensional real vector space in a 2-dimensional real vector space in $\mathbb{R}^3$ looks like a 1-dimensional flagpole attached to a 2-dimensional flag. (Remember that on the 4th of July!)

9) And now for something completely different. Let $G = SU(n)$, and let H be the center of G. (That is, the matrices in G that commute with all of G). Show that G/H is a Lie group. Is it isomorphic to a linear group? (You know it is when $n = 2$. Is it for all n?) If so, find an explicit embedding into $GL(m,\mathbb{R})$ or $GL(m,\mathbb{C})$ for some sufficiently large m.

Let $\mathbf{g} = \mathbf{su}(n) = \mathbb{R}^{n^2-1}$. G acts on on $\mathbf{g}$ by conjugation, thereby inducing a homomorphism $G \to GL(n^2-1,\mathbb{R})$. (Actually, the image is in $SO(n^2-1)$, but we don't need that fact.) The kernel of this map is precisely the center $H = Z_n$, so this factors through an injection $G/H \to GL(n^2-1,\mathbb{R})$. Thus G/H can be realized as a subgroup of $GL(n^2-1,\mathbb{R})$ – actually a subgroup of $SO(n^2-1)$. Note that $m = n^2-1$ is generally a LOT bigger than n. When $n = 2$, we have an isomorphism $G/H = SO(3)$, but for larger values of n the embedding of G/H into $SO(n^2-1)$ is very far from surjective. The (real) dimension of G/H is n^2-1, while the dimension of $SO(n^2-1)$ is $(n^2-1)(n^2-2)/2$. Even for $n = 3$ this is an embedding of an 8-dimensional group in a 28-dimensional group.