Lie Groups, Problem Set # 9
Due Thursday, November 15

1) Consider the group $G = O(2)$. Write out a set of coordinate patches for G, and construct a (non-trivial) left-invariant volume form. (The zero form doesn't count.) Then construct a right-invariant volume form. Show that these forms *cannot* be normalized to agree everywhere.

$O(2)$ consists of matrices of one of two forms:

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \qquad \text{and} \qquad T_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}.$$

If we define $r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $T_\theta = R_\theta r = r R_{-\theta}$. Note also that $r R_\theta r^{-1} = R_{-\theta}$. A left-invariant volume form is $d\theta$ on the R_θ's and $-d\theta$ on the T_θ's. A right-invariant volume form would be $d\theta$ on the R_θ's and $d\theta$ on the T_θ's. Note that these are the same on the identity component of $O(2)$ but disagree on the other component. Since $Ad(r)$ sends ∂_θ to $-\partial_\theta$, and hence $d\theta$ to $-d\theta$, there's no way to have a bi-invariant volume form.

2) Show that every real classical group is unimodular.

We already showed (problem 3.1.4 from homework #5) that the Lie algebra of every real classical group has a non-degenerate Ad-invariant bilinear form. Call this form $\phi_{\mathbf{g}}$. For each $a \in G$, $Ad(a) : \mathbf{g} \to \mathbf{g}$ is in $Aut(\phi_{\mathbf{g}})$. But this means that $Ad(a)^T \tilde{\phi}_{\mathbf{g}} Ad(a) = \tilde{\phi}_{\mathbf{g}}$, hence that $\det(Ad(a)^T)\det(\tilde{\phi}_{\mathbf{g}})\det(Ad(a)) = \det(\tilde{\phi}_{\mathbf{g}})$. Since $\phi_{\mathbf{g}}$ is non-degenerate, $\det(\tilde{\phi}_{\mathbf{g}}) \neq 0$, so $\det(Ad(a))^2 = 1$, so $|\det(Ad(a))| = 1$.

Note that this does NOT prove that $\det(Ad(a)) = +1$. We just saw a counterexample. In problem 1, $\det(Ad(r)) = -1$.

3) Show that $SL(2,\mathbb{R})$ is unimodular but does not admit a bi-invariant metric.

Since this is a classical group, it is unimodular. Let $a = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$ and let $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. If we had a bi-invariant metric, then we would have $\langle Ad(a)X | Ad(a)X \rangle = \langle X|X \rangle$. But $Ad(a)X = 4X$, so $\langle Ad(a)X|Ad(a)X\rangle = 16\langle X|X\rangle$. But an inner product must have $\langle X|X\rangle > 0$, so we have a contradiction.

Note that $sl(2,\mathbb{R})$ does have a symmetric ad-invariant bilinear form of signature $(2,1)$, namely $Tr(XY)$, which can be extended to a bi-invariant bilinear form on all of $SL(2,\mathbb{R})$. But a symmetric non-degenerate bi-linear form is not an inner product. An inner product must be positive! By contrast, $Tr(X^2) = 0$.

4) Problem 5.2.3 Note: I think he got j_ℓ and j_r mixed up in the statement of part (b). Also, we should have positive powers of α_i in the definitions of j_r and j_ℓ. State and prove the correct result. Don't worry about the correspondence with equation (7).

(a) We need to show that $d_l(a) = d_r(a) = |\det(a)|^{-n} da$, or equivalently that $da = |\det(a)|^n d_l(a) = |\det(a)|^n d_r(a)$. Let's first work with d_l. We proceed as in the proof (in class) of Weyl's Integration Formula, taking a basis for $\mathbf{g}$, mapping it to a by left translation, and evaluating it.

Let $e_1, \ldots, e_n$ be a basis for $\mathbb{R}^n$. Then $E_{ij} = e_i e_j^T$ is a basis for $\mathbf{g}$. But $aE_{ij} = (ae_i)e_j^T$. For each fixed j, the subspace spanned by the E_{ij}'s is preserved by a_ℓ, and the determinant of the action on this subspace is just $\det(a)$. Since there are n such subspaces, the determinant of the action of a_ℓ on the n^2 dimensional space of all matrices is $(\det(a))^n$, and so the volume element transforms by $|\det(a)|^n$.

The calculation for right-translation is similar, since $E_{ij}a = e_i(a^T e_j)^T$. Now it's the subspaces with a given i that are preserved by right-multiplication by a, and the action of a_r^{-1} on each one has a determinant of $\det(a^T) = \det(a)$. There are n such subspaces, so there are n powers of $|\det(a)|$ in the transformation of the volume element.

(b) Now a basis for $\mathbf{g}$ is the E_{ij}'s with $i \leq j$. The action of a_ℓ once again preserves the space spanned by the E_{ij}'s with fixed j. The action on this j-dimensional space is given by the upper left $j \times j$ block of a, whose determinant is $\alpha_1 \cdots \alpha_j$. Multiplying things out for each j we get n powers of α_1, $n-1$ of α_2, etc. In other words, $da = \prod_j |\alpha_j|^{n+1-j} d_\ell(a)$. (In Rossman's notation, that's $j_r^{-1}(a) d_\ell(a)$, so $d_\ell(a) = j_r(a)da$.)

For right-multiplication, the subspaces with i fixed are preserved, and the action is by the lower-right $n+1-i \times n+1-i$ block, with determinant $\alpha_i \cdots \alpha_n$, so $da = \prod_i |\alpha_i|^i d_r(a)$, or $d_r(a) = j_\ell(a)da$.

5) Problem 5.2.5.

The factor of $\frac{1-\cos(\|X\|)}{\|X\|^2}$ comes from the Jacobian of the exponential map. Recall (Theorem 5 on page 15) that $d\exp_X = \exp(X)\frac{1-\exp(-ad(X))}{ad(X)}$, and that the adjoint representation of $SO(3)$ is just $SO(3)$ itself. This means that the determinant of $d\exp_X$ is the determinant of $\frac{1-\exp(-X)}{X}$, since $\det(\exp(X)) = 1$.

Let $f(s) = \frac{1-e^{-s}}{s} = \sum_{k=0}^\infty (-1)^k s^k/(k+1)!$. Our determinant is the product of $f(\lambda_i)$, where λ_i range over the eigenvalues of X, namely $\pm i\|X\|$ and 0. Since $f(0) = 1$, this leaves $(1-e^{-i\|X\|})(1-e^{i\|X\|})/\|X\|^2 = 2(1-\cos(\|X\|)/\|X\|^2$.

Now for normalization. We want the integral over the ball of radius π in $\mathbf{g}$ to be 1, but $\int_{\|X\|<\pi} \frac{1-\cos(\|X\|)}{\|X\|^2} d^3X = \int_0^\pi \frac{1-\cos(r)}{r^2} 4\pi r^2 dr = 4\pi^2$. So our normalized volume element is $\frac{1}{4\pi^2} \frac{1-\cos(\|X\|)}{\|X\|^2} d^3X$, as required.

6) Problem 5.2.8

Since $SL(2,\mathbb{R})$ is unimodular, we can work with either the left-invariant measure or the right-invariant measure, since they're the same. It looks like this problem is most easily done with right-invariance.

(a) To figure out the right-invariant measure, we need to compute $(\partial_\theta a)a^{-1}$, $\partial_\sigma a a^{-1}$, and $\partial_\tau a a^{-1}$, and apply the standard volume form on $\mathbf{g}$ to the result. With

$$a(\theta,\sigma,\tau) = k(\theta)n(\sigma)a(\tau) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^\tau & 0 \\ 0 & e^{-\tau} \end{pmatrix},$$

we have

$$\partial_\theta a a^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = k(\theta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} k(-\theta),$$

$$\partial_\sigma a a^{-1} = \begin{pmatrix} -\sin(\theta)\cos(\theta) & \cos^2(\theta) \\ -\sin^2(\theta) & \sin(\theta)\cos(\theta) \end{pmatrix} = k(\theta) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} k(-\theta),$$

$$\partial_\tau a a^{-1} = k(\theta) \begin{pmatrix} 1 & -2\sigma \\ 0 & -1 \end{pmatrix} k(-\theta)$$

Since $\det(ad(k(\theta))) = 1$, feeding these three elements of $\mathbf{g}$ to the volume form is the same as feeding $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & -2\sigma \\ 0 & -1 \end{pmatrix}$, which yields a constant, independent of θ, σ and τ. In other words, the invariant measure is an (arbitrary) multiple of $d\theta d\sigma d\tau$.

(b) If we instead use the parametrization $a(\theta,\sigma,\tau) = k(\theta)a(\tau)n(\sigma)$, then

$$(\partial_\theta a)a^{-1} = k(\theta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} k(-\theta),$$

$$(\partial_\tau a)a^{-1} = k(\theta) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} k(-\theta),$$

$$(\partial_\sigma a)a^{-1} = k(\theta) \begin{pmatrix} 0 & e^{2\tau} \\ 0 & 0 \end{pmatrix} k(-\theta),$$

so our invariant measure is a multiple of $e^{2\tau} d\theta d\sigma d\tau$.