M408R Final Exam, December 12, 2014

1) Plug-and-chug grab bag (2 pages): Each part is worth 4 points.

a) Compute the derivative of $x^3 \ln(x)$.

b) Compute the derivative of $\dfrac{xe^x}{\sin(x)}$

c) Compute the derivative of x^e.

d) Find all solutions to $10^{(x^2)} = 1,000,000,000$.

e) Simplify: $2\log_3(45) - \log_3(25)$.

f) Evaluate: $\int_1^2 6x^2 - 8x + 2dx$.

g) Evaluate: $\int (3x^2 + 7)\ln(x)dx$.

h) Evaluate: $\int_0^1 6x\sqrt{1-x^2}dx$.

i) Evaluate: $\dfrac{d}{dx}\displaystyle\int_3^x \frac{\ln(t+5)}{1+\tan^{-1}(3t)}dt$

j) Evaluate: $\displaystyle\int \frac{xe^x dx}{(x+1)^2}$. [Hint: Integration by parts with $u = xe^x$ works. More obvious choices don't.]

2) Derivatives. Suppose that $f(x)$ and $g(x)$ are differentiable functions, that $g(1) = 3$, $g'(1) = 2$, $f(3) = 5$, and $f'(3) = -1/4$. Let $h(x) = f(g(x))$.

a) Using the microscope equation for g, find the approximate value of $g(1.06)$. Call this number a.

b) Using the microscope equation for f, find the approximate value of $f(a)$.

c) Find $h'(1)$.

d) Use the microscope equation for h to find the approximate value of $h(1.06)$.

3) A Thanksgiving turkey comes out of the oven when the meat is 170° (Fahrenheit). It cools at a rate proportional to the difference between the turkey's temperature and room temperature (which we'll take to be 70°).

a) Write down a *differential (aka rate) equation* that describes this cooling process. Be sure to explain what each of your variables means, and what each of your parameters mean.

b) Now solve the differential equation. What is the temperature of the turkey as a function of time? (This answer will still involve parameter(s).)

c) Suppose that after 10 minutes the turkey has cooled to 140°. Using that information, find the values of the parameter(s) in your differential equation. This should give you the temperature as an explicit function of time.

d) Following up on part (c), after how much time will the turkey be 100° (and ready to carve and eat)?

4) (2 pages) Charged particles in magnetic fields tend to move along helices centered on the field lines. This is what causes the Aurora Borealis. The following problem explores the math behind behind this phenomenon.

Let $v_x(t)$, $v_y(t)$ and $v_z(t)$ be the components of an electron's velocity in the x, y and z directions at time t. If the magnetic field points in the z direction, then the laws of physics say these variables will satisfy the following rate equations:

$$\begin{aligned} v_x' &= cv_y \\ v_y' &= -cv_x \\ v_z' &= 0, \end{aligned}$$

where c is a parameter that depends on the strength of the magnet.

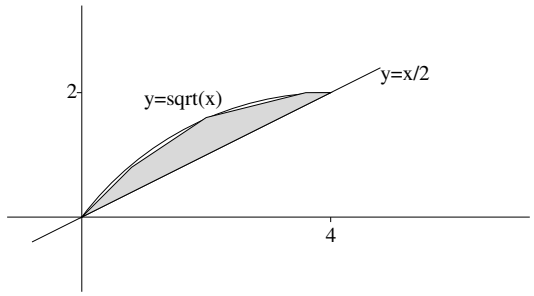
a) Suppose that $c = 1/2$, that $v_x(0) = 10$, $v_y(0) = 20$, and $v_z(0) = 15$. Use Euler's method, with step size $h = 0.2$, to estimate $v_x(0.4)$, $v_y(0.4)$, and $v_z(0.4)$. (Along the way you'll also compute $v_x(0.2)$, $v_y(0.2)$ and $v_z(0.2)$.)

b) The exact solution to the initial value problem is:

$$\begin{aligned} v_x(t) &= 10\cos(t/2) + 20\sin(t/2), \\ v_y(t) &= 20\cos(t/2) - 10\sin(t/2), \\ v_z(t) &= 15. \end{aligned}$$

Now that we know the components of the velocity, we can infer the coordinates $x(t)$, $y(t)$ and $z(t)$ of the position. Assuming that $x(0) = y(0) = z(0) = 0$, find $x(t)$, $y(t)$, and $z(t)$ (exactly).

5) Vertical and horizontal slices. We wish to determine the area of the shaded region in the figure below, between the curve $y = \sqrt{x}$ and $y = x/2$.



a) Express this area as an integral over x. (That is, slice it into vertical slices). Be clear about what function(s) you are integrating, and about the limits of integration. Do not evaluate the integral (yet).

b) Express this area as an integral over y. (That is, slice it into horizontal slices). Be clear about what function(s) you are integrating, and about the limits of integration. Do not evaluate the integral (yet).

c) Evaluate one of these two integrals (your choice) to get the area. Your final answer should be a number, like 3 or 17 or $5\sqrt{2}$.

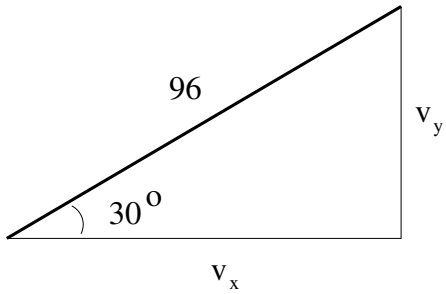
6) Growth spurts. According to a certain model (that I just made up), teenage boys grow at an average rate of $12te^{-t}$ inches/year, where t is the age in years minus 13. (That is $t = 3$ means "16 years old".)

a) Write down an integral that gives the average height gain between a boy's 14th and 18th birthdays. (You do not need to evaluate the integral.)

b) If a boy is 4 foot 10 inches high on his 13th birthday, how tall can he expect to get by the time he is 17? Note that this part uses different start and end times than part (a). Your answer should be a precise number, like "5 foot 4.5 inches" (not the correct answer, BTW).

7) (2 pages) This problem walks you through the following physics problem: "A quarterback throws a pass at 96ft/sec aimed 30° above horizontal. Ignoring air resistance, how far will the ball fly before being caught (at the same height as it was thrown)?"

a) If a ball is moving with speed 96 feet/sec in a direction 30° above horizontal, what are the x and y components of its velocity? Call these $v_x(0)$ and $v_y(0)$, respectively. (This isn't really a calculus problem. It's a trig problem.)



b) For now, ignore the horizontal part of the motion. We're just going to study the up-and-down motion. If the vertical acceleration is $a_y(t) = -32$ feet/sec^2 and the initial vertical velocity is given by your answer to part (a), what is the vertical velocity $v_y(t)$ for all time?

c) If the vertical velocity is given by your answer to (b) and the initial height is $y(0) = 6$ (the height of the quarterback), what is $y(t)$ for all time?

d) At what positive time t is $y(t)$ again equal to 6 (the height of the receiver)? (That's the time at which the ball is caught.)

e) In the mean time, the horizontal velocity has not changed at all. If $x(0) = 0$, what is x when the ball is caught?