M408M Final Exam Solutions, December 14, 2006

1. Conic sections and polar coordinates. Consider the plane curve

$$r = \frac{10}{1+\sin(\theta)}.$$

a) Is this a circle, ellipse, parabola or hyperbola?

It is a parabola. If you don't see this from the polar form of the equation, see part (c).

b) Find the (Cartesian) coordinates of the point closest to the origin.

"Closest the the origin" means the smallest value of r, which means the biggest denominator, which means $\theta = \pi/2$. In polar coordinates our point is $(5, \pi/2)$, while in Cartesian coordinates the point is (0,5).

c) Express the equation of the curve in Cartesian coordinates.

Multiplying both sides by $1 + \sin(\theta)$ gives $10 = r + r\sin(\theta) = r + y$, so $r = 10 - y$, so $r^2 = 100 - 20y + y^2$. But $r^2 = x^2 + y^2$, so $x^2 = 100 - 20y$, or $y = 5 - (x^2/20)$.

2. Parametrized curves. Consider the parametrized plane curve $x(t) = 2e^t$, $y(t) = t - e^{2t}/2$.

a) Find the arclenth of this curve between $t = -2$ and $t = 2$.

Since $dx/dt = 2e^t$ and $dy/dt = (a - e^{2t})$, we have $(dx/dt)^2 + (dy/dt)^2 = 4e^{2t} + 1 - 2e^{2t} + e^{4t} = (1 + e^{2t})^2$, so our arclength is $\int_{-2}^{2} 1 + e^{2t} dt = 4 + (e^4 - e^{-4})/2$.

b) Find the slope of the tangent line to the curve at $t = 0$.

$dy/dx = (dy/dt)/(dx/dt) = (1 - e^{2t})/2e^t = 0/2 = 0$. The curve is horizontal at this point.

3. Lines and planes in 3D space.

a) Find the equation (in standard form) of the line through the points $(1, 1, 0)$ and $(3, 0, 1)$.

The vector along the line is $(3, 0, 1) - (1, 1, 0) =< 2, -1, 1 >$, so our line is $\frac{x-3}{2} = -y = z - 1$, or equivalently $\frac{x-1}{2} = 1 - y = z$ or equivalently $< x, y, z >=< 1, 1, 0 > + < 2, -1, 1 > t$. Any of these answers were worth full credit.

b) Find the equation of the plane containing this line and the point (2,2,1).

First we need to find the vector perpendicular to the plane. This is the cross product of $< 2, -1, 1 >$ and $(2, 2, 1) - (1, 1, 0) =< 1, 1, 1 >$. This is $< -2, -1, 3 >$, so our plane is $-2(x - 1) - (y - 1) + 3z = 0$, or equivalently

$2x + y - 3z = 3$.

c) Find the equation of the line through (1,1,0) that is perpendicular to the plane you found in part (b).

$\frac{x-1}{-2} = \frac{y-1}{-1} = z/3$.

4. Position, velocity and acceleration.

A particle is moving with acceleration $\vec{a}(t) =< 3\cos(t), 4\sin(t), 0 >$. Its velocity at time zero is $\vec{v}(0) =< 0, 0, 12 >$ and its position at time zero is $\vec{r}(0) =< 2, 1, 4 >$.

a) Find the velocity $\vec{v}(t)$ as a function of time.

Integrate the acceleration to get the velocity, using the initial condition to set the constants of integration. $\vec{v}(t) =< 3\sin(t), 4 - 4\cos(t), 12 >$.

b) Find the position $\vec{r}(t)$ as a function of time.

Integrate the velocity to get the position, using the initial condition to set the constants of integration. $\vec{r}(t) =< 5 - 3\cos(t), 4t - 4\sin(t) + 1, 12t + 4 >$.

c) Find the speed of the particle at time $t = \pi/2$.

Since $\vec{v}(\pi/2) =< 3, 4, 12 >$, the speed is $|\vec{v}| = \sqrt{3^2 + 4^2 + 12^2} = 13$.

Extra Credit) Find the average velocity between times $t = 0$ and $t = \pi$.

This is $(\vec{r}(\pi) - \vec{r}(0))/\pi =< \frac{6}{\pi}, 4, 12 >$.

5. Surfaces.

Consider the surface $x^4 + y^4 + z^2 = 26$, which passes through the point $(1, 2, 3)$.

a) Find a (nonzero) vector normal to this surface at $(1, 2, 3)$.

Let $g(x, y, z) = x^4 + y^4 + z^2$. A normal vector is $\nabla g =< 4x^3, 4y^3, 2z >=< 4, 32, 6 >$. Any multiple of this, like $< 2, 16, 3 >$ is equally good.

b) Find the equation of the tangent plane to the surface at $(1, 2, 3)$.

Since $< 2, 16, 3 >$ is a normal vector, our plane is $2x + 16y + 3z = 43$.

c) Use this tangent plane to estimate the value of y when $x = 1.01$ and $z = 3.004$. (Actually, there are TWO values of y on the surface. Pick the positive one.)

Plugging $x = 1.01$ and $z = 3.004$ into the equation of the plane gives $y = 1.998$ (exactly).

Another way of doing the problem is to compute $\partial y/\partial x$ and $\partial y/\partial z$ using implicit differentiation. It gives the same final answer, of course.

6. Max-min. Find all critical points of the function $f(x, y) = x^4 + y^4 - 4xy + 57$. Which of these are local maxima? Which

are local minima? Which are saddle points? [Note: this problem is straight from the text]

See page 991 of the text.

$\nabla f =< 4x^3 - 4y, 4y^3 - 4x >$. Setting this equal to zero gives $y = x^3$ and $x = y^3$, hence $x = x^9$, whose solutions are $(x, y) = (-1, -1), (0, 0), (1, 1)$. Since $f_{xx} = 12x^2$ and $f_{yy} = 12y^2$ and $f_{xy} = -4$, we apply the second derivative test at each point to see that $(1, 1)$ and $(-1, -1)$ are local minima, while $(0, 0)$ is a saddle point.

7. The concentration of radioactivity in an area is given by the function $f(x, y) = x^3 + y^3 - 4xy^2 + 84$.

a) Find the gradient of this function at the point $(1, 1)$.

$\nabla f =< 3x^2 - 4y^2, 3y^2 - 8xy >=< -1, -5 >$.

b) Find the directional derivative of this function in the direction of the point $(4, 5)$.

The displacement is $(4, 5) - (1, 1) = (3, 4)$, whose unit direction vector is $\vec{u} = (3/5, 4/5)$. The directional derivative is $\nabla f \cdot \vec{u} = -23/5$.

c) If we head northwest from $(1, 1)$ (where the y axis points north and the x axis points east) with speed $5\sqrt{2}$, at what rate will the function be changing?

In this case our velocity is $\vec{v} = (-5, 5)$, and $df/dt = \vec{v} \cdot \nabla f = -20$.

8. Laminates

A rectangular laminate lies in the $x - y$ plane, with vertices at $(-1, -2)$, $(-1, 2)$, $(1, -2)$ and $(1, 2)$. The density of the laminate at the point (x, y) is $(1 + x)y^2$.

a) Find the mass of the laminate.

$M = \int_{x=-1}^{1} \int_{y=-2}^{2} (1 + x)y^2 dy\, dx = 32/3$.

b) Find the center of mass of the laminate.

$M_x = \int_{-1}^{1} \int_{-2}^{2} (x + x^2)y^2 dy\, dx = 32/9$, $M_y = \int_{-1}^{1} \int_{-2}^{2} (1 + x)y^3 dy\, dx = 0$, so $\bar{x} = M_x/M = 1/3$, $\bar{y} = M_y/M = 0$, and our center of mass is at $(1/3, 0)$.

9. Consider the iterated integral $\int_{x=0}^{1} \int_{y=x}^{1} e^{y^2} dydx$.

a) Sketch the region of integration.

This is a triangle with vertices at $(0, 0)$, $(0, 1)$ and $(1, 1)$.

b) Rewrite the double integral over this region as an iterated integral where you integrate first over x and then over y. Clearly indicate your limits of integration.

$\int_{y=0}^{1} \int_{x=0}^{y} e^{y^2} dx\, dy$

c) Evaluate the resulting iterated integral.

$= \int_0^1 ye^{y^2} dy = \frac{1}{2}e^{y^2}\Big|_0^1 = \frac{e-1}{2}$.

10. Consider the dome-shaped region above the plane $z = 0$ and below the parabaloid $z = 1 - x^2 - y^2$. Call this region R.

a) Write $\int\int\int_R e^{-z} dV$ as an iterated integral. You are free to do this in Cartesian, cylindrical or spherical coordinates, or something even stranger, and you may integrate over your variables in whatever order you wish. However, you MUST make clear your order of integration and limits of integration, and express the integrand (and if necessary, the Jacobian) in the coordinates you have chosen.

There are many possible answers. My favorite is $\int_{r=0}^1 \int_{\theta=0}^{2\pi} \int_{z=0}^{1-r^2} e^{-z} r dz d\theta dr$.

b) Evaluate the integral. [N.B. For part (a), all coordinate systems are equally good and will get full credit, as long as you set up the integral correctly. However, part (b) will be a lot easier if you make a sensible choice in part (a).]

Integrating over z gives $\int_0^1 \int_0^{2\pi} r(1 - e^{r^2-1}) d\theta\, dr = 2\pi \int_0^1 r(1 - e^{r^2-1}) dr$. Using the substitution $u = r^2 - 1$, this evaluates to $\pi(u - e^u)|_{-1}^0 = \pi/e$.