M408M Final Exam, December 14, 2006

1. Conic sections and polar coordinates. Consider the plane curve

$$r = \frac{10}{1 + \sin(\theta)}.$$

a) Is this a circle, ellipse, parabola or hyperbola?

b) Find the (Cartesian) coordinates of the point closest to the origin.

c) Express the equation of the curve in Cartesian coordinates.

2. Parametrized curves. Consider the parametrized plane curve $x(t) = 2e^t$, $y(t) = t - e^{2t}/2$.

a) Find the arclenth of this curve between $t = -2$ and $t = 2$.

b) Find the slope of the tangent line to the curve at $t = 0$.

3. Lines and planes in 3D space.

a) Find the equation (in standard form) of the line through the points $(1, 1, 0)$ and $(3, 0, 1)$.

b) Find the equation of the plane containing this line and the point (2,2,1).

c) Find the equation of the line through (1,1,0) that is perpendicular to the plane you found in part (b).

4. Position, velocity and acceleration.

A particle is moving with acceleration $\vec{a}(t) =< 3\cos(t), 4\sin(t), 0 >$. Its velocity at time zero is $\vec{v}(0) =< 0, 0, 12 >$ and its position at time zero is $\vec{r}(0) =< 2, 1, 4 >$.

a) Find the velocity $\vec{v}(t)$ as a function of time.

b) Find the position $\vec{r}(t)$ as a function of time.

c) Find the speed of the particle at time $t = \pi/2$.

Extra Credit) Find the average velocity between times $t = 0$ and $t = \pi$.

5. Surfaces.

Consider the surface $x^4 + y^4 + z^2 = 26$, which passes through the point $(1, 2, 3)$.

a) Find a (nonzero) vector normal to this surface at $(1, 2, 3)$.

b) Find the equation of the tangent plane to the surface at $(1, 2, 3)$.

c) Use this tangent plane to estimate the value of y when $x = 1.01$ and $z = 3.004$. (Actually, there are TWO values of y on the surface. Pick the positive one.)

6. Max-min. Find all critical points of the function $f(x, y) = x^4 + y^4 - 4xy + 57$. Which of these are local maxima? Which are local minima? Which are saddle points? [Note: this problem is straight from the text]

7. The concentration of radioactivity in an area is given by the function $f(x, y) = x^3 + y^3 - 4xy^2 + 84$.

a) Find the gradient of this function at the point $(1, 1)$.

b) Find the directional derivative of this function in the direction of the point $(4, 5)$.

c) If we head northwest from $(1, 1)$ (where the y axis points north and the x axis points east) with speed $5\sqrt{2}$, at what rate will the function be changing?

8. Laminates

A rectangular laminate lies in the $x - y$ plane, with vertices at $(-1, -2)$, $(-1, 2)$, $(1, -2)$ and $(1, 2)$. The density of the laminate at the point (x, y) is $(1 + x)y^2$.

a) Find the mass of the laminate.

b) Find the center of mass of the laminate.

9. Consider the iterated integral $\int_{x=0}^{1} \int_{y=x}^{1} e^{y^2} dydx$.

a) Sketch the region of integration.

b) Rewrite the double integral over this region as an iterated integral where you integrate first over x and then over y. Clearly indicate your limits of integration.

c) Evaluate the resulting iterated integral.

10. Consider the dome-shaped region above the plane $z = 0$ and below the parabaloid $z = 1 - x^2 - y^2$. Call this region R.

a) Write $\int\int\int_R e^{-z} dV$ as an iterated integral. You are free to do this in Cartesian, cylindrical or spherical coordinates, or something even stranger, and you may integrate over your variables in whatever order you wish. However, you MUST make clear your order of integration and limits of integration, and express the integrand (and if necessary, the Jacobian) in the coordinates you have chosen.

b) Evaluate the integral. [N.B. For part (a), all coordinate systems are equally good and will get full credit, as long as you set up the integral correctly. However, part (b) will be a lot easier if you make a sensible choice in part (a).]