M408D Second Midterm Exam Solutions, November 5, 2002

1. Planes:

a) Find the equation of the plane that goes through the point (2,1,0) and has normal vector (2,1,-2).

Let $\mathbf{N} = (2, 1, -2)$. The equation of the plane is $\mathbf{N} \cdot \mathbf{x} = \mathbf{N} \cdot (2, 1, 0)$, or $2x + y - 2z = 5$.

b) Find the distance from the point (4,13,5) to the plane in part (a).

The distance is $\mathbf{N} \cdot [(4, 13, 5) - (2, 1, 0)]/|N| = 6/3 = 2$.

c) Find the equation of the plane that goes through the three points (2,1,0), (3,1,4) and (0,0,0) [No, this isn't the same plane as part a)].

Since one of the points is the origin, the normal vector to this plane is $\mathbf{N}' = (2, 1, 0) \times (3, 1, 4) = (4, -8, -1)$, so the plane is $4x - 8y - z = 0$.

d) Find the cosine of the angle between the planes of part (a) and (c).

This is $|\mathbf{N} \cdot \mathbf{N}'|/|N||N'| = |2|/(3 \times 9) = 2/27$.

2. Lines

a) Find the equation, in symmetric form, for the line through the point (2,1,0) in the direction (2,1,-2).

The direction vector is $d = (2, 1, -2)$, the base point is $P_0 = (2, 1, 0)$, so the line is

$$\frac{x-2}{2} = \frac{y-1}{1} = \frac{z}{-2}.$$

b) How far is the origin (0,0,0) from this line?

The vector from P_0 to the origin is $v = (-2, -1, 0)$. The distance is $|v \times d|/|d| = |(2, -4, 0)|/|(2, 1, -2)| = 2\sqrt{5}/3$.

c) Find the equation, in symmetric form, for the line through the two points (2,1,0) and (3,-1,4).

The direction vector is $d' = (3, -1, 4) - (2, 1, 0) = (1, -2, 4)$, so our line is

$$\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z}{4}.$$

You could also write

$$\frac{x-3}{1} = \frac{y+1}{-2} = \frac{z-4}{4}.$$

d) Find the equation (in symmetric form) of the line through (2,1,0) that is perpendicular to the lines of parts (a) and (c).

The vector normal to d and d' is $d \times d' = (0, -10, -5)$. Rescale it to $(0, 2, 1)$. Our line is

$$x = 2, \qquad \frac{y-1}{2} = \frac{z}{1}.$$

(If you don't rescale the direction vector, then you get an uglier, but completely equivalent, answer.)

3. Parametrized curves: Consider the parametrized curve

$$\mathbf{r}(t) = \left(\frac{t^2}{2}, \frac{4}{3}t^{3/2}, 2t + 5 \right).$$

a) Compute the position, velocity, unit tangent vector and speed at time $t = 1$.

$\mathbf{r}'(t) = (t, 2t^{1/2}, 2)$, so position $= \mathbf{r}(1) = (1/2, 4/3, 7)$, velocity $= \mathbf{r}'(1) = (1, 2, 2)$, speed $= |(1, 2, 2)| = 3$, and unit tangent vector $= \mathbf{r}'(1)/|\mathbf{r}'(1)| = (1/3, 2/3, 2/3)$.

b) Compute the arc-length of the curve from $t = 0$ to $t = 2$.

Note that the speed is $|\mathbf{r}'(t)| = \sqrt{t^2 + 4t + 4} = t + 2$, so the arclength is $\int_0^2 (t + 2) dt = 6$.

4. Polar coordinates.

a) Sketch the curve $r = 1 + \cos(2\theta)$. Mark clearly the angles (if any) where the curve goes through the origin, and the angles where r is maximal.

There are two lobes, one to the right and one to the left. The curve goes through the origin at $\theta = \pi/2$ and $\theta = 3\pi/2$. The farthest points are when $\cos(2\theta) = 1$, so $2\theta = 2n\pi$, so $\theta = n\pi$. That is, the positive and negative x directions.

b) Find (in polar coordinates!) the points where this curve intersects the circle $r = 1/2$.

Note that r is never negative, so we don't have to worry about "accidental" intersections. Set $1/2 = 1 + \cos(2\theta)$, so $\cos(2\theta) = -1/2$, to $2\theta = \pm 2\pi/3 + 2n\pi$, so $\theta = \pm\pi/3 + n\pi$. For $\theta \in [0, 2\pi)$, these are the points $(r, \theta) = (1/2, \pi/3), (1/2, 2\pi/3), (1/2, 4\pi/3), (1/2, 5\pi/3)$.

c) Write down a definite integral that gives the area, in the first quadrant, inside the curve $1 + \cos(2\theta)$ but outside the circle $r = 1/2$.

$$\int_0^{\pi/3} \frac{(1 + \cos(2\theta))^2 - (1/2)^2}{2} d\theta.$$

Extra credit: Evaluate this integral. Expand it out and use the double-angle formula $\cos^2(2\theta) = (1 + \cos(4\theta))/4$ to convert the integral to

$$\int_0^{\pi/3} \frac{5}{8} + \cos(2\theta) + \frac{\cos(4\theta)}{4} d\theta = \frac{5\pi}{24} + \frac{7\sqrt{3}}{32}.$$

5. Partial derivatives. Consider the function of two variables $F(x, y) = x^3y + 2x + y^2$.

a) Compute $\partial F/\partial x$ and $\partial F/\partial y$ and evaluate these at the point (1,1).

$\partial F/\partial x = 3x^2y+2$, evaluated at (1,1) gives 5. $\partial F/\partial y = x^3+2y$, evaluated at (1,1) gives 3.

b) Use these to estimate the value of $F(1.01, 1)$ and the value of $F(1, 1.01)$.

The change in F from (1,1) to (1.01,1) is roughly $(0.01)\partial F/\partial x = 0.05$, so $F(1.01, 1) \approx 4.05$. The change in F from (1,1) to (1,1.01) is roughly $(0.01)\partial F/\partial y = 0.03$, so $F(1.01, 1) \approx 4.03$.

c) Estimate the value of $F(1.02, 1.01)$.

Add the effect of changing x to the effect of changing y: $4 + 5(0.02) + 3(0.01) = 4.13$.

d) Compute the second-order partial derivatives $\partial^2F/\partial x^2$, $\partial^2F/\partial y^2$, and $\partial^2F/\partial x\partial y$.

In order, the answers are $6xy$, 2, and $3x^2$.