M408M Final Exam, December 14, 2013

1. Lines and planes. (16 pts) Let $P(1,0,1)$, $Q(0,1,2)$, $R(-1,-1,1)$, and $S(0,0,10)$ be points in $\mathbb{R}^3$.

a) Let L be the line through P and Q. Express the equation of L in either vector or parametric form (your choice), and then express the equation in symmetric form.

b) Find the equation of the plane through P, Q and R.

c) Find the distance from S to the plane you found in part (b).

d) Find the distance from R to the line L.

2. Parametric curves (12 pts)

a) First consider the parametric curve $x = 2t - 2\sin(t)$, $y = 3 - 2\cos(t)$ in $\mathbb{R}^2$. Find the slope of the line tangent to this curve at $t = \pi/4$ (i.e. tangent at the point $(\pi/2 - \sqrt{2}, 3 - \sqrt{2})$).

b) Now consider the curve $\mathbf{r}(t) = \langle 2t - 2\sin(t), 3 - 2\cos(t), 4t\rangle$. If this is the trajectory of a particle, find the velocity and acceleration as a function of time.

c) Find the equation of the line tangent to this 3D curve at $t = \pi/4$. (You can express your answer in your choice of vector, parametric, or symmetric form.)

3. Polar coordinates. (16 pts) Consider the polar curve $r = e^\theta$, where θ runs from 0 to 2π.

a) Find the slope of this curve at $\theta = \pi/6$.

b) Find the arc-length of this curve.

c) Let R be the region bounded by the positive x axis, the positive y axis, and this curve. Find the area of R.

d) Compute $\iint_R 3re^{-3\theta} dA$.

4. Partial derivatives and gradients. (16 pts) Let $f(x, y, z) = y^2 + xy + (y + 1)\ln(z) + 3y$.

a) Compute the gradient of $f(x, y, z)$ as a function of (x, y, z).

b) Compute all the second order partial derivatives. (There are 9 of these, but $f_{xy} = f_{yx}$, etc, leaving six calculations.)

c) If $\mathbf{r}(t) = \langle\cos(t), \sin(t), 1 + t\rangle$ is a parametrized curve, compute $\frac{df(\mathbf{r}(t))}{dt}$ at $t = 0$.

d) Let S be the surface $f(x, y, z) = 0$. Find the equation of the plane tangent to S at $(1, 0, 1)$.

5. Maxima and minima. (12 pts) Note that parts (a) and (b) have nothing to do with parts (c) and (d). Parts (c) and (d) ask you to solve the same problem in two different ways. To get credit for a part, you **must** solve it using the method indicated.

a) Find all local extrema of the function $h(x, y) = x^3 - 12xy + 8y^3$.

b) For each critical point, indicate whether it is a local maximum, a local minimum, or a saddle point.

c) **Using Lagrange multipliers**, find the maximum and minimum values of $f(x, y) = xy$ on the ellipse $\frac{x^2}{4} + y^2 = 1$.

d) Now write the ellipse as a **parametrized curve** and express $f(x, y)$ as a function of t. Then use 1-dimensional calculus to find the maximum and minimum values of f.

6. (4 pts) Let R be the region in the plane between the parabola $y = x^2$ and the line $y = 2x$. Compute $\iint_R x^2 + 2y dA$.

7. A trickier double integral. (12 pts) Consider the iterated integral

$$\int_0^1 \int_{x^2}^1 \sin(\pi y^{3/2}) dy dx.$$

a) Draw the region of integration. (In other words, rewrite the iterated integral as a double integral $\iint_R$ (some function)dA and draw a picture of R.)

b) Rewrite the double integral as an iterated integral where we first integrate over x and then over y. Be careful with your limits of integration!

c) Evaluate this new-and-improved iterated integral.

8. Change of variables. (12 pts) Let R be the diamond-shaped region with corners at $P(1, 0)$, $Q(2, -1)$, $S(3, 0)$ and $T(2, 1)$. We are trying to evaluate the double integral $\iint_R e^{x+2y} dA$.

a) Write x and y as functions of new parameters u and v, so that we are at P when $u = v = 0$, Q when $u = 1$ and $v = 0$, T when $v = 1$ and $u = 0$, and S when $u = v = 1$. In other words, find a mapping that sends the unit square to R.

b) Find the Jacobian of this mapping.

c) Rewrite the double integral as an integral over u and v.

d) Evaluate the double integral.