M408M Third Midterm Exam Solutions, November 21, 2013

1a) Compute the gradient of the function $f(x, y, z) = xe^y + yz^2 + xz$.

$$\nabla f = \langle e^y + z, xe^y + z^2, 2yz + x\rangle.$$

b) Let S be the surface $xe^y + yz^2 + xz = 3$. Compute the equation of the plane tangent to S at the point $P(1, 0, 2)$.

The normal to this plane is the gradient of f evaluated at P, namely $\langle 3, 5, 1\rangle$. This means that the plane must be of the form $3x + 5y + z = D$ for some number D. Plugging in P gives our answer: $3x + 5y + z = 5$.

c) Staying on the surface S, estimate the value of y when $x = 1.01$ and $z = 2.02$.

Plugging in the values of x and z into the equation of the tangent plane gives $y = -0.01$. Equivalently, the tangent plane says that $3dx+5dy+dz = 0$. Since $dx = 0.01$ and $dz = 0.02$, dy must be -0.01.

2. I want to build a storage shed in my back yard with a square footprint. That's four walls (all of the same size) and a square roof (no floor). The materials for the walls cost \$1 per square foot, while the materials for the roof cost \$4 per square foot. What are the dimensions of the shed of maximum volume that I can build for \$108?

Call the width (and depth) of the shed x, and the height z. The volume is then $f(x, z) = x^2z$. Since there are 4 sides of area xz and a roof of area x^2, the cost of the shed is $g(x, z) = 4xz + 4x^2$. We are maximizing f subject to the constraint $g = 108$.

Since $\nabla f = \langle 2xz, x^2\rangle$ and $\nabla g = \langle 8x + 4z, 4x\rangle$, our Lagrange multiplier equations are:

$$\begin{aligned} 2xz &= \lambda(8x + 4z) \\ x^2 &= \lambda 4x \\ 4x^2 + 4xz &= 108 \end{aligned}$$

From the second equation we get $\lambda = x/4$ (or $x = 0$, but that doesn't make any sense.) Plugging into the first equation gives $z = 2x$. Plugging that into the third equation gives $12x^2 = 108$, so $x = 3$. So our shed is 3 feet wide, 3 feet deep, and 6 feet high.

3. For each of the following functions, find the critical point(s) and determine which are maxima, which are minima, which are saddles, and which are something else.

a) $f(x,y) = xy - x - y - x^2 - y^2$

$\nabla f = \langle y - 1 - 2x, x - 1 - 2y \rangle$. Setting this equal to zero gives the two equations

$$\begin{aligned} -2x + y &= 1 \\ x - 2y &= 1 \end{aligned}$$

whose unique solution is $x = y = -1$. So $(-1,-1)$ is our only critical point.

The second derivatives are $f_{xx} = -2$, $f_{yy} = -2$ and $f_{xy} = 1$. Since $f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} < 0$, $(-1,-1)$ is a local maximum.

b) $f(x,y) = e^{x^2-y^2}$.

Now the gradient is $\langle 2xe^{x^2-y^2}, -2ye^{x^2-y^2} \rangle$. Setting this equal to zero gives $x = y = 0$ as our only critical point. Then we compute

$$\begin{aligned} f_{xx} &= (2 + 4x^2)e^{x^2-y^2} \\ f_{xy} &= -4xye^{x^2-y^2} \\ f_{yy} &= (-2 + 4y^2)e^{x^2-y^2} \end{aligned}$$

Evaluating at $(0,0)$ gives $f_{xx} = 2$, $f_{xy} = 0$ and $f_{yy} = -2$. Since the discriminant is negative, this is a saddle point.

4. a) Compute the iterated integral

$$\int_0^2 \int_0^x 6ye^{(x^3)} dy dx.$$

First we integrate over y: $\int_0^x 6ye^{(x^3)} dy = 3y^2 e^{(x^3)}|_0^x = 3x^2 e^{(x^3)}$. Our outer integral is then

$$\int_0^2 3x^2 e^{(x^3)} dx = e^{(x^3)}|_0^2 = e^8 - 1.$$

b) We want to compute the double integral $\iint_R f(x,y)dA$, where $f(x,y) = \ln(e^{xy}+7)$ and R is the region between the curve $y = e^x$ and the line $y = 1 + \frac{e^2-1}{2}x$. Express this double integral as an **iterated integral** where we integrate first over y and then over x. Be clear about what the limits of integation are for both the x-integral and the y-integral. (For heavens sake, **do not attempt** to evaluate the iterated integral. It's a mess.)

First note that the line and the exponential meet at $(0,1)$ and $(2,e^2)$.

The bottom of each column is at $y = e^x$, and the top is at $y = 1 + \frac{e^2-1}{2}x$. The left-most column is at $x = 0$ and the right-most column is at $x = 2$, so our integral is

$$\int_{x=0}^{2} \int_{y=e^x}^{1+\frac{e^2-1}{2}x} \ln(e^{xy}+7)dydx.$$

c) Now set up (but do not evaluate!) another iterated integral that computes $\iint_R f(x,y)dA$ where now we integrate first over x and then over y.

Now we're organizing our boxes into rows. The left endpoint of each row is on the line, whose equation is transformed to $x = \frac{2(y-1)}{e^2-1}$, and the right endpoint is on the curve $x = \ln(y)$. The bottom row is at $y = 1$ and the top row is at $y = e^2$, so our integral is

$$\int_{y=1}^{e^2} \int_{x=\frac{2(y-1)}{e^2-1}}^{\ln(y)} \ln(e^{xy}+7)dxdy.$$

The most common mistakes on this problem were
a) Writing functions instead of numbers on the outer integral. Outer limits are **always** numbers, since they mark the maximum/minimum height of a row or the leftmost/rightmost position of a column.
b) Writing numbers instead of functions in the inner integral. That would make the region of integration a rectangle (which it isn't).
c) Swapping the upper and lower limits of integration for the inner integral of part (c). If you draw a picture of the region, you'll see that the line is on the **left** and the curve is on the **right**.