M408N First Midterm Exam Solutions, September 20, 2012

1) (15 pts) Suppose that at a certain time there are 500 bacteria growing in a Petri dish. The population grows exponentially, doubling every hour.

a) Find a formula for the number $x(t)$ of bacteria t hours later.

$x(t) = 500 \cdot 2^t$, since we start with 500 and double every hour.

b) Find a formula for t in terms of x.

$2^t = x/500$, so $t = \log_2(x/500)$. You could also write $\log_2(x) - \log_2(500)$ or $\log_2(x) - 3\log_2(5) - 2$.

c) Now suppose that the bacteria double every 20 minutes (instead of every hour). How does this change the answers to parts (a) and (b)?

In that case we would have $x(t) = 500 \cdot 2^{3t} = 500 \cdot 8^t$ and $t = \log_2(x/500)/3$ (or $\log_8(x/500)$).

2. (15 points) Compute the following quantities exactly. The answers may involve square roots, in which case you can leave your answers looking like $\sqrt{3}/7$ or $5\sqrt{2}$ (which aren't actually the answers, of course).

a) Draw a right triangle where one of the angles has a tangent of 2. Mark the lengths of the three sides clearly. Then compute the sine of that angle.

The triangle I had in mind has opposite side of length 2 and adjacent side of length 1 (that is, vertices at (0,0), (1,0) and (1,2)). The hypotenuse is then $\sqrt{2^2+1^2} = \sqrt{5}$, and the sine is opposite/hypotenuse $= 2/\sqrt{5}$ (or $2\sqrt{5}/5$).

b) Now draw a right triangle involving an angle whose sine is 1/2. Mark the lengths of all three sides, and compute the secant of that angle.

Now the opposite side has height 1/2 and the hypotenuse has length 1, so the adjacent side has length $\sqrt{1-(1/2)^2} = \sqrt{3}/2$. The secant is hypotenuse/adjacent $= 2/\sqrt{3}$.

c) Compute $\sin(\cos^{-1}(1/3))$.

This is the sine of an angle whose cosine is 1/3. Draw a triangle with adjacent side 1/3 and hypotenuse 1, hence opposite side of length $\sqrt{8/9}$, so the sine of the angle is $\sqrt{8/9} = 2\sqrt{2}/3$.

3. (10 pts) Consider the function $f(x) = \begin{cases} 2x+1 & x<2 \\ 4 & x=2 \\ 7-x & x>2 \end{cases}$

a) Compute $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2} f(x)$, if they exist.

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 7-x = 5$, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2x+1 = 5$. Since both 1-sided limits give 5, $\lim_{x\to 2} f(x) = 5$.

b) Is $f(x)$ continuous everywhere? Why or why not?

$f(x)$ is NOT continuous, since $5 = \lim_{x\to 2} f(x) \neq f(2) = 4$. The function has a removable discontinuity at $x = 2$.

4. (30 pts) Compute the following limits:

a) $\lim_{x\to -1} \dfrac{x^2-x-2}{x-2}$.

Both the numerator and denominator are continuous, and the denominator doesn't go to 0 as $x \to -1$, so this function is continuous at $x = -1$. Plug in $x = -1$ to get $0/(-3) = 0$.

b) $\lim_{x\to 2} \dfrac{x^2-x-2}{x-2}$.

The numerator is $(x-2)(x+1)$, so we have $\lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2} x+1 = 3$.

c) $\lim_{x\to 1^+} \dfrac{x}{1-x}$.

When x is slightly bigger than 1, the denominator is small and negative, while the numerator is close to 1, so the ratio is huge and negative. This makes the limit $-\infty$.

d) $\lim_{x\to (\frac{\pi}{2})^+} \sin(x)\tan(x)$.

First, expand $\sin(x)\tan(x) = \sin^2(x)/\cos(x)$. When x is a little bigger than $\pi/2$, $\cos(x)$ is slightly negative, while $\sin^2(x)$ is approximately 1. This is just like (c), with a limit of $-\infty$.

e) $\lim_{x\to\infty} \dfrac{2x^3-x^2+17x-5}{3x^3+139x^2-47x+\pi}$

Divide the top and the bottom by x^3 to get $\lim_{x\to\infty} \frac{2-x^{-1}+17x^{-2}-5x^{-3}}{3+139x^{-1}-47x^{-2}+\pi x^{-3}} = 2/3$.

f) $\lim_{x\to -\infty} \dfrac{|x^5+3|}{x^4+2x^2+1}$.

The numerator grows faster than the denominator, so the limit is either

$\pm\infty$. To see which, imagine plugging in a large negative value of x. The numerator is positive (being an absolute value), as is the denominator (which is x^4 plus change), so the ratio is positive and growing. The limit is ∞.

5. (30 points) True or False (no partial credit, and no penalty for guessing)

a) If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, then $\lim_{x\to a} f(x)$ exists.

FALSE. The one-sided limits might be different. (E.g., $\lim_{x\to 0} |x|/x$.)

b) If $f(x)$ is a polynomial, then $\lim_{x\to a} f(x) = f(a)$.

TRUE. All polynomials are continuous.

c) The statment $\lim_{x\to\infty} f(x) = -\infty$ means that whenever x is sufficiently large and positive, $f(x)$ is large and negative.

TRUE. (The precise definition requires us to say what "large" means.)

d) If $f(x)$ and $g(x)$ are continuous at $x = a$, then so are $f(x)+g(x)$, $f(x)g(x)$, and $f(x)/g(x)$.

FALSE. For $f(x)/g(x)$ to be continuous, or even defined, we also need $g(a) \neq 0$.

e) $\ln(75e^2) - 2\ln(5) - \ln(3) = 2$.

TRUE. $\ln(75e^2) = \ln(3 \cdot 5^2 \cdot e^2) = \ln(3) + 2\ln(5) + 2$.

f) The inverse function of $f(x) = 3e^x + 1$ is $f^{-1}(x) = \log_e(\frac{x-1}{3})$.

TRUE. If $y = 3e^x + 1$, then $e^x = (y-1)/3$ and $x = \ln((y-1)/3)$. (And $\ln = \log_e$)

g) For every x where both sides are defined, $\cot^2(x) + 1 = \sec^2(x)$.

FALSE. $\cot^2(x) + 1 = \csc^2(x)$, not $\sec^2(x)$.

h) If $f(x)$ is continuous on the interval $[0, 4]$, then $\lim_{x\to 3} f(x)$ must exist.

TRUE. Not only that, but the limit must equal $f(3)$.

i) $\log_{10}(e) = \log_e(10)$.

FALSE. These numbers are reciprocals. Since $10 > e$, $\log_e(10) > 1$ while $\log_{10}(e) < 1$.

j) $\log_{32}(2) = 1/5$.

TRUE. $32 = 2^5$, so $2 = 32^{1/5}$.