1) (15 pts) (Inverse) trig functions

a) Draw a right triangle where one of the angles is $\tan^{-1}(2)$. (There are many possible answers, all with the same shape but different overall size. Pick your favorite.) Label the lengths of all three sides and then compute $\sin(\tan^{-1}(2))$.

One choice for this triangle would have opposite side 2, adjacent side $1$, and hypotenuse $\sqrt{2^2 + 1^2} = \sqrt{5}$. By soh-cah-toa, the sine of $\tan^{-1}(2)$ is $\frac{2}{\sqrt{5}}$.

b) Compute $\cos\left(\frac{5\pi}{6}\right)$.

This is one of the angles that you should have memorized. $5\pi/6$ radians is 150 degrees, the sine is 1/2 and the cosine is $-\sqrt{3}/2$.

c) Draw a right triangle where one of the angles is $\sec^{-1}(2)$. Label the lengths of all three sides and then compute $\tan(\sec^{-1}(2))$.

This triangle has hypotenuse 2, adjacent side 1, and so opposite side $\sqrt{2^2 - 1^2} = \sqrt{3}$. The tangent of $\sec^{-1}(2)$ is then $\sqrt{3}/1 = \sqrt{3}$. You could also get this from the formula $\tan^2(\theta) = \sec^2(\theta) - 1 = 3$. By the way, the angle $\sec^{-1}(2)$ is $\pi/6$, or 30 degrees.

2. (20 points) Compute the following limits:

a) $\lim_{x \to 4} \frac{x^2 - 5x + 4}{x^2 - 16} = \lim_{x \to 4} \frac{(x - 1)(x - 4)}{(x + 4)(x - 4)} = \lim_{x \to 4} \frac{x - 1}{x + 4} = \frac{3}{8}$.

b) $\lim_{x \to \infty} \frac{x^2 - 5x + e^{-x}}{x^2 - 16 + 3e^{-x}} = \lim_{x \to \infty} \frac{1 - 5/x + e^{-x}/x^2}{1 - 16/x^2 + 3e^{-x}/x^2} = \frac{1}{1} = 1$. This is because, as $x \to \infty$, $1/x$, $1/x^2$ and $e^{-x}/x^2$ all go to zero.

c) $\lim_{x \to -\infty} \frac{x^2 - 5x + e^{-x}}{x^2 - 16 + 3e^{-x}}$.

In this direction, $e^{-x}$ does NOT go to zero. Rather, it is the dominant term, growing (much) faster than all the other terms, so the limiting ratio is 1/3.

d) $\lim_{x \to 0^+} \frac{\tan(3x)}{4x}$.

Rewrite this as $\frac{\sin(3x)}{4x\cos(4x)}$. As $x \to 0$, $\sin(3x)/3x \to 1$, so $\sin(3x)/4x \to 3/4$, while $\cos(4x) \to \cos(0) = 1$, so the limit is 3/4.
3. (15 pts) Continuity and discontinuities.

a) Where does the function \( f(x) = \frac{x^2 - 9}{x^2 - 4x + 3} \) fail to be continuous?

*The ratio of polynomials is continuous everywhere except where the denominator is zero. Since the denominator factors as \((x - 3)(x - 1)\), the discontinuities are at \( x = 1 \) and \( x = 3 \).*

b) For each point where \( f(x) \) isn’t continuous, identify the kind of discontinuity.

*The numerator factors as \((x - 3)(x + 3)\). The discontinuity at \( x = 3 \) is removable, since the limit as \( x \to 3 \) exist (and equals 3), insofar as the \((x - 3)\)'s in the numerator and denominator cancel. However, the function has an infinite discontinuity at \( x = 1 \), since the denominator goes to zero while the numerator does not.*

4. (15 pts) Definition of derivative.

Consider the limit \( \lim_{h \to 0} \frac{(5 + h)e^{5+h} - 5e^5}{h} \).

a) Find a function \( f(x) \) and a point \( a \) such that this limit equals \( f'(a) \).

*There’s actually more than one answer, but the simplest one is \( f(x) = xe^x \) and \( a = 5 \). Then we’re taking the limit of \((f(a+h) - f(a))/h\), which is \( f'(a) \).*

b) Using what you know about taking derivatives, evaluate the limit.

*By the product rule, \( f'(x) = e^x + xe^x = (x+1)e^x \), so \( f'(5) = 6e^5 \).*

5. (20 pts) Compute the derivatives of the following functions with respect to \( x \).

a) \((\sin(x) + 3)(e^x + x^2)\).

*By the product rule, this is \((\cos(x))(e^x + x^2) + (\sin(x) + 3)(e^x + 2x)\).*

b) \(\frac{\sin(x) + 3}{e^x + x^2}\).

*By the quotient rule, this is \(\frac{[\sin(x)+3](e^x+2x) - (\cos(x))(e^x+x^2)}{(e^x+x^2)^2}\).*

c) \(\sin(e^{5x} + x^2)\).

*By the chain rule (applied twice), this is \(\cos(e^{5x} + x^2)(5e^{5x} + 2x)\).*
d) \( \sin^2(x) + \cos^2(x) \).

There are two ways to see that this is zero. One is to recognize that \( \sin^2(x) + \cos^2(x) = 1 \). The other is to just use the chain rule or product rule on each term. The derivative is \( 2 \sin(x) \cos(x) + 2 \cos(x)(-\sin(x)) = 0 \).

6. (15 pts) Implicit differentiation.

Find the equation of the line tangent to the curve \( x^2 + y^3 = 9 \) at the point \((1, 2)\).

We have \( 2x + 3y^2 y' = 0 \), so \( y' = \frac{-2x}{3y^2} \). At \((1, 2)\) this gives \( y' = -2/12 = -1/6 \). By the point-slope formula, the equation of our line is then \( (y - 2) = -(x - 1)/6 \), or equivalently \( y = \frac{13 - x}{6} \), or equivalently \( 6y + x = 13 \).