M408N Final Exam Solutions, December 12, 2015

1) Grab bag (50 points). Compute the following quantities:

a) $f'(x)$, where $f(x) = (x^2 + 3x)\ln(x)$.

By the product rule, this is $\frac{x^2+3x}{x} + (2x+3)\ln(x) = x + 3 + (2x+3)\ln(x)$.

b) $\dfrac{dg}{dt}$, where $g(t) = \dfrac{\cos(t)}{t^2+1}$.

By the quotient rule, this is $\dfrac{-(t^2+1)\sin(t) - 2t\cos(t)}{t^2+1}$.

c) The derivative of $\sin(\ln(x^2+2))$ with respect to x.

By the chain rule, applied twice, this is $\cos(\ln(x^2+2))\dfrac{2x}{x^2+2}$.

d) $\displaystyle\lim_{x\to 3} \frac{\frac{1}{3} - \frac{1}{x}}{x-3}$.

Since $\frac{1}{3} - \frac{1}{x} = \frac{x-3}{3x}$, this reduces to

$$\lim_{x\to 3} \frac{(x-3)/3x}{x-3} = \lim_{x\to 3} \frac{1}{3x} = \frac{1}{9}.$$

You could also get this answer by applying L'Hospital's rule.

e) $\displaystyle\lim_{x\to 1} \frac{2e^{x-1} - 2}{\ln(x)}$.

Since both the numerator and denominator go to zero, we can use L'Hospital's rule. The limit equals $\displaystyle\lim_{x\to 1} \frac{2e^{x-1}}{1/x} = \frac{2}{1} = 2$.

f) $\displaystyle\lim_{\theta\to\frac{\pi}{2}} (\sec(\theta) - \tan(\theta))$.

Rewrite the quantity as $\dfrac{1}{\cos(\theta)} - \dfrac{\sin(\theta)}{\cos(\theta)} = \dfrac{1-\sin(\theta)}{\cos(\theta)}$. Both the numerator and denominator go to zero as $\theta \to \pi/2$, so we can use L'Hospital's rule:

$$\lim_{\theta\to\frac{\pi}{2}} \frac{1-\sin(\theta)}{\cos(\theta)} = \lim_{\theta\to\frac{\pi}{2}} \frac{-\cos(\theta)}{-\sin(\theta)} = \frac{0}{-1} = 0.$$

g) $\dfrac{d(x^{1/x})}{dx}$.

If $f(x) = x^{1/x}$, then $\ln(f(x)) = \ln(x)/x$, so

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}\frac{\ln(x)}{x} = \frac{1-\ln(x)}{x^2}.$$

But then $f'(x) = \frac{1-\ln(x)}{x^2} x^{1/x}$.

h) $\int_2^3 3x^2 - 4x + 1\, dx$.

By FTC2, this is $x^3 - 2x^2 + x\Big|_2^3 = 12 - 2 = 10$.

i) $\displaystyle \frac{d}{dx} \int_{-2}^{x} \frac{t^2 e^{-t} dt}{1+t^4}$.

By FTC1, this is $\displaystyle \frac{x^2 e^{-x}}{1+x^4}$.

j) $\displaystyle \frac{d}{dt} \int_{-3t^2}^{e^t} \cos(se^s) ds$.

The integral is $F(e^t) - F(-3t^2)$, where $F'(s) = \cos(se^s)$, so the derivative of the integral is

$$e^t F'(e^t) + 6tF'(-3t^2) = e^t \cos(e^t e^{(e^t)}) + 6t\cos(-3t^2 e^{-3t^2}).$$

2. Continuity and differentiability. Consider the function

$$f(x) = \begin{cases} x & x < -1 \\ 0 & x = -1 \\ \frac{x^2+x}{x+1} & -1 < x < 0 \\ -2x & 0 \le x < 1 \\ \frac{x}{x-2} - 1 & 1 \le x < 2 \\ 42 & x = 2 \\ \frac{x}{x-2} - 1 & x > 2 \end{cases}$$

This function is obviously continuous and differentiable away from the four points (-1, 0, 1, and 2) where the formula changes. This question is about what happens at those four points.

a) List all points where the function is discontinuous? (There may be more than one.)

At $x = -1$, the limits from both sides are -1, but $f(-1) = 0$.

At $x = 0$ the limit from both sides is 0, as is $f(0)$.

At $x = 1$ the limit from both sides is -2, as is $f(1)$.

At $x = 2$ the limit from the left is $-\infty$, from the right is $+\infty$, and the value is 42.

Bottom line: the function is discontinuous at -1 and 2, but continuous at 0 and $+1$.

b) For each of these points, indicate what kind of discontinuity the function has.

At $x = -1$ we have a removable discontinuity. The overall limit does exist, but doesn't equal the value of the function.

At $x = 2$ we have an infinite discontinuity, since the limits on the two sides are $\pm\infty$.

c) List all the points at which $f(x)$ is continuous but not differentiable.

Next we check what happens around $x = 0$ and $x = 1$. At $x = 0$, the slope goes from 1 to -2, so there is no overall limit of $(f(h) - f(0))/h$, and the function is not differentiable.

At $x = 1$, the slope is -2 on both sides, as can be seen by taking the derivatives of $-2x$ and $\frac{x}{x-2} - 1$. So the function IS differentiable there.

3. Tangent lines and linear approximations. Consider the function
$f(x) = \dfrac{e^{x-2}}{x}$.

a) Find the equation of the line tangent to $y = f(x)$ at $(2, f(2))$.

We compute:

$$f(2) = \frac{1}{2}; \qquad f'(x) = \frac{xe^{x-2} - e^{x-2}}{x^2}; \qquad f'(2) = \frac{1}{4},$$

so our line is $y - \frac{1}{2} = \frac{1}{4}(x - 2)$, or equivalently $y = \frac{x}{4}$.

b) Use this line to approximate $f(2.04)$.

Plugging in $x = 2.04$ gives $y - \frac{1}{2} = 0.01$, so $y = 0.51$, so $f(2.04) \approx 0.51$.

4. Local maxima and minima. Consider the function

$$f(x) = e^x(\sin(x) - \cos(x)).$$

a) Find all the critical points of $f(x)$ on the interval $[-4, 8]$.

We compute $f'(x) = 2e^x \sin(x)$ (from the product rule). This is zero precisely where $\sin(x) = 0$, since e^x is always positive. That is, when x is a multiple of π. Between -4 and 8 this occurs at:

$$x \in \{-\pi, 0, \pi, 2\pi\}.$$

b) Use the second derivative test to determine which of these are local maxima and which are local minima. (To get full credit, you MUST use the second derivative test.)

$f''(x) = 2e^x(\sin(x) + \cos(x))$. Plugging in at the four points, we see that $f''(x) > 0$ at $x = 0$ and $x = 2\pi$, but that $f''(x) < 0$ at $x = \pm\pi$. Thus there are local maxima at $x = \pm\pi$ and local minima at $x = 0$ and $x = 2\pi$.

5. Anti-derivatives. A block is moving along a 1-dimensional track with acceleration $a(t) = 12 - 6t$. At time $t = 0$, its velocity is $v(0) = -9$ and its position is $x(0) = 13$.

a) Find the velocity $v(t)$ as a function of time.

The velocity is the anti-derivative of acceleration, so $v(t) = 12t - 3t^2 + c_1$. Since $v(0) = -9$, we must have $c_1 = -9$, so $v(t) = -3t^2 + 12t - 9$. Note that this factors as $v(t) = -3(t - 1)(t - 3)$.

b) Find the position $x(t)$ as a function of time.

The position is the anti-derivative of velocity, so $x(t) = -t^3 + 6t^2 - 9t + c_2$. Since $x(0) = 13$, we must have $c_2 = 13$, so $x(t) = -t^3 + 6t^2 - 9t + 13$.

c) At what times is the block moving forward?

In other words, when is the velocity positive? This happens when $1 < t < 3$.

d) At what times is the velocity increasing?

In other words, when is the acceleration positive? This happens when $t < 2$.

6. Definite integrals and Riemann sums.

a) Approximate the integral $\int_{-2}^{4} \ln(x + 3)dx$ as a Riemann sum with 6 terms, using right endpoints. [You can leave your answer in terms of logs, but you should make each term explicit. That is, you might write something like "$(\ln(20) + \ln(22) + \ln(24) + \ln(26) + \ln(28) + \ln(30))/4$", but not something like $\sum \ln(x_k + 3)\Delta x$.]

We break the interval $[-2, 4]$ into 6 pieces, with $x_0 = -2$, $x_1 = -1$, all the way to $x_6 = 4$. Since $\Delta x = (4-(-2))/6 = 1$, our sum is $f(x_1)+\cdots+f(x_6) = \ln(2) + \ln(3) + \ln(4) + \ln(5) + \ln(6) + \ln(7)$ (which equals $\ln(7!)$, by the way). If we had been using left endpoints instead of right, we would have gotten $f(x_0) + \cdots + f(x_5) = \ln(1) + \cdots + \ln(6) = \ln(6!)$.

b) Now approximate the integral $\int_{-2}^{4} \ln(x+3)dx$ as a Riemann sum with N terms, using right endpoints. Leave your answer **in Σ notation**.

Now we have $\Delta x = 6/N$, so $x_k = -2 + 6k/N$, so $f(x_k) = \ln(1 + 6k/N)$, so our Riemann sum is

$$\sum_{k=1}^{N} f(x_k)\Delta x = \frac{6}{N}\sum_{k=1}^{N} \ln(1 + \frac{6k}{N}).$$

c) Compute $\lim_{N\to\infty} \frac{6}{N}\sum_{k=1}^{N} 3\left(-2 + \frac{6k}{N}\right)^2$ by converting it to an integral and then using the Fundamental Theorem of Calculus.

This is also an integral from -2 to 4, only of the function $g(x) = 3x^2$, so the limit equals

$$\int_{-2}^{4} 3x^2 dx = x^3\Big|_{-2}^{4} = 64 - (-8) = 72.$$