M408N Third Midterm Exam Solutions, November 21, 2016

1. (2 pages, 34 points) Derivatives and curves. Consider the function $f(x) = x^4 - 4x^3 + 4x^2$.

a) Make a sign chart for f. Where is $f(x)$ positive? Negative? Zero?

The function factors as $x^2(x-2)^2$, so it is zero at $x = 0$ and $x = 2$ and positive everywhere else. Note that this gives us absolute (and hence local) minima at $x = 0$ and $x = 2$, although you didn't need to notice that at this point to solve the problem.

b) Make a sign chart for f'. What are the critical points? Identify which critical points are local maxima, which are local minima, and which are neither.

$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-1)(x-2)$. The critical points are 0, 1 and 2. Note that $f'(x)$ is negative for $x < 0$, positive for $0 < x < 1$, negative for $1 < x < 2$ and positive for $x > 2$. By the first derivative test, there are local minima at $x = 0$ and $x = 2$ (we already knew that!) and a local maximum at $x = 1$. You could also apply the second derivative test to get the same classification.

c) Make a sign chart for f''. [Note: this is the only place in the problem where numbers that aren't integers might appear.]

$f''(x) = 12x^2 - 24x + 8 = 4(3x^2 - 6x + 2)$. Applying the quadratic formula says that $f''(x) = 0$ when $x = 1 \pm \frac{\sqrt{3}}{3}$. $f''(x)$ is positive for $x < 1 - \frac{\sqrt{3}}{3}$, negative for $1 - \frac{\sqrt{3}}{3} < x < 1 + \frac{\sqrt{3}}{3}$ and positive for $x > 1 + \frac{\sqrt{3}}{3}$.

(Incidentally, $1 - \sqrt{3}/3 \approx 0.423$ and $1 + \sqrt{3}/3 \approx 1.577$, so the inflection points are roughly half-way between the critical points.)

d) Sketch the curve $y = f(x)$. It doesn't have to be precise, but it should be positive/negative in the right places, should be increasing/decreasing in the right places, should be concave up/down in the right places, and should have the correct asymptotes (if any).

This graph is U-shaped on $(-\infty, 1 - \sqrt{3}/3)$, with a local minimum at the origin, then looks like an upside-down U on $(1 - \sqrt{3}/3, 1 + \sqrt{3}/3)$, with a local maximum at $(1, 1)$, and then U-shaped on $(1 + \sqrt{3}/3, \infty)$, with a local minimum at $(2, 0)$.

2. (2 pages, 36 points) NASA is designing a rectangular solar panel for a mission to Mars. Because of the bizarre (and totally made up) constraints of space travel, a vertical strut of height y meters requires e^y kg of material, while a horizontal strut of length x meters requires x kg of material. A panel includes two vertical struts (left and right) and two horizontal struts (top and bottom), and we can only bring 12 kg of building material. What are the dimensions of the panel that will maximize the area? We'll tackle this step by step.

a) Draw a picture. Let x be the width of the panel and y the height.

The picture is just a rectangle with an "x" on the bottom and a "y" on a side. The area is xy, of course.

b) What is the relation between x and y? [Remember that we are using a total of 12kg of building material]

The total amount of material needed is $2x + 2e^y = 12$, so $x + e^y = 6$, or $x = 6 - e^y$.

c) Eliminate x, and express the area in terms of y only.

The area is $A(y) = xy = y(6 - e^y) = 6y - ye^y$.

d) Take the derivative with respect to y and set it equal to zero to get an equation for y.

$0 = A'(y) = 6 - (y + 1)e^y$.

e) This equation cannot be solved exactly! Instead, get an approximate solution by starting with an initial guess $y = 1$, and running **one** iteration of Newton's method.

Let $f(y) = 6 - (y + 1)e^y$. We are trying to solve $f(y) = 0$, with initial guess $y_0 = 1$. Since $f'(y) = -(y + 2)e^y$, we have

$$y_1 = y_0 - \frac{f(y_0)}{f'(y_0)} = 1 - \frac{6 - 2e}{-3e} = \frac{6 + e}{3e} \approx 1.069.$$

(No, you weren't expected to do the decimal approximation!)

3. (2 pages, 30 points) Indeterminate forms and antiderivatives.

a) Compute $\lim_{x\to 3} \frac{x^3-27}{x^2-9}$

This is a standard "0/0" limit, so L'Hospital's rule applies. We have

$$\lim_{x\to 3} \frac{x^3-27}{x^2-9} = \lim_{x\to 3} \frac{3x^2}{2x} = \frac{9}{2}.$$

b) Compute $\lim_{x\to 0} \frac{\ln(\cos(x))}{\sin^2(x)}$

This is also a "0/0" limit, so L'Hospital tells us that

$$\lim_{x\to 0} \frac{\ln(\cos(x))}{\sin^2(x)} = \lim_{x\to 0} \frac{-\sin(x)/\cos(x)}{2\sin(x)\cos(x)} = \lim_{x\to 0} \frac{-1}{2\cos^2(x)} = -1/2.$$

(Instead of factoring the $\sin(x)$ from the numerator and denominator, you could also apply L'Hospital's rule a second time to get the same answer, but that's more work.)

c) Compute $\lim_{x\to 0} (\cos(x))^{\csc^2(x)}$

The log of this limit is PRECISELY the limit you just computed in part (b), so the answer is $e^{-1/2}$.

d) Find the most general antiderivative of $f(x) = \cos(x) + e^{2x} + (x-1)^3$.

This is $\sin(x) + \frac{e^{2x}}{2} + \frac{(x-1)^4}{4} + C$.

e) Find $f(x)$ if $f'(x) = \sec^2(x) + \frac{1}{(x+1)^2}$ and $f(0) = 5$.

$f(x) = \tan(x) - \frac{1}{x+1} + C$. Setting $f(0) = 5$ gives $C = 6$, so

$$f(x) = \tan(x) - \frac{1}{x+1} + 6.$$