M408R Makeup First Exam Solutions

1) Let $P(t)$ be the number of patients in a certain hospital on day t, and suppose that the rate equation for P is

$$P' = 20 - \frac{1}{8}P$$

and that there are 200 patients on October 7.

a) Use Euler's method with step size $h = 2$ to estimate the number of patients on October 9.

At time $t = 7$, we have $P = 200$, so $P'(7) = 20 - 200/8 = -5$. So with $h = 2$ we have $P(9) = P(7+h) \approx P(7) + hP'(7) = 200 + 2(-5) = 190$.

b) Use Euler's method with step size $h = 2$ to estimate the number of patients on October 5.

$P(5) = P(7-h) \approx P(y) - hP'(7) = 200 - 2(-5) = 210.$

c) Use Euler's method with step size $h = 1$ to estimate the number of patients on October 9.

As before, we have $P'(7) = -5$, so $P(8) \approx 200 + 1(-5) = 195$. Then $P'(8) = 20 - P(8)/8 = -4.375$, so $P(9) \approx 195 + 1(-4.375) = 190.625$.

2) Here is a table of values of a function $f(x)$.

x	$f(x)$
9.96	0.99825933842
9.97	0.99869515831
9.98	0.99913054128
9.99	0.99956548822
10	1.00000000000
10.01	1.00043407748
10.02	1.00086772153
10.03	1.00130093302
10.04	1.00173371281

a) Find $f'(10)$ to within 0.0001.

If you use forward differences, the best estimate is $[f(10.01)-f(10)]/0.01 = 0.043407748$. If you use backward differences, the best estimate is $[f(10) - f(9.99)]/0.01 = 0.043451178$. With centered differences, the best estimate is $[f(10.01) - f(9.99)]/0.02 = 0.0434295$. These are all within 0.0001 of 0.0434 (and all will be accepted as correct). For what it's worth, the above function happens to be $f(x) = \log_{10}(x)$, and $f'(10) = 1/(10\ln(10)) = 0.04342944819$.

b) Use this information and the microscope equation to estimate $f(10.25)$.

$f(10.25) \approx f(10)+0.25f'(10) \approx 1.01086$. *The actual value of* $\log_{10}(10.25)$ *is slightly lower, namely* 1.010724.

3) Suppose that $f(x)$ is a differentiable function with $f(5) = 18$ and $f'(5) = 3$.

a) Find an equation for the line tangent to $y = f(x)$ at $(5, 18)$.

Since the slope is 3 *and it goes through* $(5, 18)$, *the line is*

$$y = 18 + 3(x - 5) = 3x + 3.$$

b) Estimate the values of $f(5.03)$ and $f(4.92)$.

$f(5.03) \approx f(5)+.03f'(5) = 18.09$, *and* $f(4.92) \approx f(5)-0.08f'(5) = 17.76$.

4) Ozone is an unstable form of oxygen, with molecular formula O_3. Let $C(t)$ be the concentration of ozone in the upper atmosphere at time t. Ozone is created when cosmic rays hit the upper atmosphere. Ozone is destroyed whenever two ozone molecules collide. [Note: in reality there are other ways to destroy ozone, but we'll ignore those in this problem.]

a) How does the rate of ozone creation depend on C? (E.g. is it independent of C? Proportional to C? Proportional to C^2? Some other expression?)

The rate of ozone creation is independent of how much ozone is already there, since it's a function purely of the cosmic ray count. That's a constant, which we can call a.

b) How does the rate of ozone destruction depend on C (Constant? Proportional to C? Proportional to C^2? Other?)

This is proportional to C^2. *For each ozone molecule, the decay rate is proportional to the number of other ozone molecules that could be bumped into. Multiplying by the number of ozone molecules that might decay, the total decay rate goes as* C^2. *"Goes as" means we need a proportionality constant, which we can call* b.

This is very much like the infection rate in the SIR model. The rate at which infected people sneeze on susceptible people is proportional to SI. *The rate at which ozone molecules bump into ozone molecules goes as* $CC = C^2$.

c) Write down a rate equation for C. This rate equation should involve two unknown parameters, one having to do with ozone creation and one with ozone destruction.

Since ozone is being created at rate a *and destroyed at rate* bC^2, *we must* *have*

$$C' = a - bC^2.$$

5) a) Find the derivative of the function $f(t) = 3t^2 - 12t + 14$. You may use the formulas from Section 3.5.

By the power law, $f'(t) = 6t - 12$.

b) The position of a particle is given by $x(t) = 3t^2 - 12t + 14$, where t is measured in seconds and x is measured in feet. How fast is the particle moving at time $t = 1$? Is it moving forwards or backwards?

The velocity at time $t = 1$ *is* $x'(1) = 6(1) - 12 = -6$. *This means the particle is moving* **backwards** *at 6 feet/second.*

c) At what time(s) is the particle's velocity equal to zero?

Setting $0 = 6t - 12$, *we get* $6t = 12$, *or* $t = 2$.