1. Related rates.

Consider the curve \( y^2 = x^3 + 1, \ y \geq 0 \).

a) Find the slope of the line that is tangent to the curve at the point (2,3).

Take the derivative of the equation with respect to \( x \): \( 2yy' = 3x^2 \), so \( y' = 3x^2/(2y) = 12/6 = 2 \).

b) A particle is moving along the curve. Its \( x \)-coordinate is increasing at a rate of 10 units/second. How fast is \( y \) changing when \( (x,y) = (2,3) \)?

There are two reasonably easy solutions. One is to use the result from (a): \( dy/dt = (dy/dx)(dx/dt) = 2(10) = 20 \) units/second.

The other method is to start from scratch, and take the derivative of the equation with respect to \( t \):

\[
2y \frac{dy}{dt} = 3x^2 \frac{dx}{dt}.
\]

Plugging in values of \( x, y \) and \( dx/dt \) gives \( 6(dy/dt) = 120 \), so \( dy/dt = 20 \), as before.

2. L'Hopital's Rule

Evaluate the following limits:

a) \( \lim_{x \to \infty} \frac{15x^2 - 9}{x^3 + 3x^2 + 5} = \lim_{x \to \infty} \frac{30x}{3x^2 + 6x} = \lim_{x \to \infty} \frac{30}{6x} = 0 \).

b) \( \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{2x}{1} = 6 \).

c) \( \lim_{x \to 2} \frac{\ln(x) - \ln(2)}{x - 2} = \lim_{x \to 2} \frac{1/x}{1} = \frac{1}{2} \).

d) \( \lim_{x \to 0} \frac{e^x - 1}{x^2 + 1} = 0 \). L'Hopital’s rule does not apply here.

3. Elasticity of Demand

The demand \( x \) for a new toy depends on its price \( p \) via the demand equation

\( x = 1000e^{-p} \).

a) Compute the elasticity of demand \( E(p) \) as a function of \( p \).
\[ E(p) = \frac{p \frac{dx}{dp}}{x} = \frac{p(-1000e^{-p})}{1000e^{-p}} = -p. \]

b) For what values of \( p \) is the demand elastic? For what values of \( p \) is the demand inelastic?

When \( p > 1 \), \( E < -1 \) and the system is elastic. [Under these circumstances we should lower the price to increase revenue.]

When \( p < 1 \), \( E > -1 \) and the system is inelastic. [To raise revenue, raise the price].

c) What value of \( p \) will maximize revenue?

\[ p = 1. \]

**Problem 4. Horse sense**

For the first two years of life, a pony’s height \( H(t) \) grows at a rate

\[ H'(t) = 15 - 3t^2, \]

(where height is measured in inches and time in years). At age 1, the pony is 45 inches tall.

a) How tall was the pony at birth?

\[ H(t) = \int H'(t) \, dt = \int (15 - 3t^2) \, dt = 15t - t^3 + C. \]

To evaluate the constant, use the fact that \( H(1) = 45 \), so \( 45 = 15 - 1 + C \), so \( C = 31 \). Now plug back in to get

\[ H(t) = 15t - t^3 + 31. \]

So when \( t \) was zero, \( H \) was 31.

b) How tall will the pony be at age 2?

\[ H(2) = 15(2) - 2^3 + 31 = 53 \text{ inches}. \]

**Problem 5. Indefinite integrals.**

Evaluate the following integrals:

a) \[ \int (2x + e^x) \, dx = x^2 + e^x + C \]

b) \[ \int xe^{x^2} \, dx = \frac{1}{2}e^{x^2} + C. \] (Integrate by substitution with \( u = x^2 \).)
c) \[ \int \frac{\ln(x)}{x} \, dx = \frac{(\ln(x))^2}{2} + C. \] (Integrate by substitution with \( u = \ln(x) \).)

d) \[ \int (2x + 1)^4 \, dx = \frac{(2x + 1)^5}{10} + C. \] (Integrate by substitution with \( u = 2x + 1 \).)

**Problem 6. Area under a curve.**

We are interested (OK, OK, your instructor is interested) in finding the area under the curve \( y = 2x^2 + 1 \) between \( x = 1 \) and \( x = 4 \).

a) Estimate this area using 3 rectangles. Your final answer should be an explicit number, like 13 or 152.

Each rectangle has width \((4-1)/3 = 1\). The three rectangles have height \( f(2), f(3) \) and \( f(4) \), so the estimated total area is \( f(2) + f(3) + f(4) = 9 + 19 + 33 = 61 \). [If you used the function values at 1, 2 and 3 instead of 2, 3, and 4, I gave full credit. The answer then would be 31]

b) Estimate the area using \( N \) rectangles. You can leave your answer as a sum, like \( \sum_{k=1}^{N} 4(\ln(N) - 3)/N \) (no, that’s not the right answer). Everything in the sum needs to be clearly defined, but **YOU DO NOT NEED TO SIMPLIFY OR EVALUATE THE SUM.**

\( \Delta x = (4 - 1)/N = 3/N \) and \( a = 1 \), so \( x_k = a + k\Delta x = 1 + \frac{3k}{N} \). Thus \( f(x_k) = 2(1 + \frac{3k}{N})^2 + 1 \), and our estimated area, \( \sum_{k=1}^{N} f(x_k)\Delta x \), works out to

\[ \sum_{k=1}^{N} \left( 2 \left( 1 + \frac{3k}{N} \right)^2 + 1 \right) \frac{3}{N} \]