Directions: Do all problems and show your work clearly. Put a box around your final answer for each problem. You must show your work and justify your answers to receive credit. No books, notes, or calculators are permitted, except one 8.5 x 11 sheet of notepaper with your own notes. Use the space provided; continue on the back if you need more space.

There are ten questions, each worth 10 points; 100 points total.

1. Use row reduction to find all solutions of the following system. Express your answer in vector parametric form.

\[
\begin{align*}
2x + 2z &= 4 \\
y + 2z + w &= 4 \\
x + z + w &= 4.
\end{align*}
\]

2. Let \(W\) be the subspace of \(\mathbb{R}^4\) spanned by the vectors \((1, 0, 1, 0), (2, 1, -1, 0), \) and \((1, 2, -5, 0)\).
   (a) Find a basis for \(W\).
   (b) What is the dimension of \(W\)?
   (c) Let \(x = (6, 2, 0, 0)\). Find \([x]_B\), where \(B\) is the basis found in part (a).

3. Let 
   \[
   A = \begin{bmatrix}
   10 & 20 & 30 \\
   6 & 12 & 24 \\
   12 & 15 & 1994
   \end{bmatrix}.
   \]
   (a) Find the determinant of \(A\).
   (b) Find the determinant of \(2A\).

4. Let 
   \[
   A = \begin{bmatrix}
   1 & 2 & 0 \\
   1 & 3 & 0 \\
   0 & 0 & 3
   \end{bmatrix}.
   \]
   (a) Find \(A^{-1}\).
   (b) Use your answer to (a) to solve \(Ax = (1, 2, 5)\) for \(x\).

5. Let 
   \[
   A = \begin{bmatrix}
   1 & 0 & 1 & 0 \\
   2 & 1 & -1 & 0 \\
   1 & 2 & -5 & 0
   \end{bmatrix}.
   \]
   (a) Find bases for the row space, column space, and null space of \(A\).
   (b) Define rank. What is the rank of \(A\)?

6. Let \(A\) be an \(n \times n\) matrix. List at least ten different conditions that are all equivalent to the statement “\(A\) is invertible”. (Do not include this statement itself as one of the ten.)

   (Scoring: 1 point each; 10 points maximum.)

7. Let 
   \[
   A = \begin{bmatrix}
   1 & 2 \\
   -1 & 4
   \end{bmatrix}.
   \]
   (a) Find the eigenvalues of \(A\).
   (b) For each eigenvalue, find a corresponding eigenvector.
   (c) Find a diagonal matrix \(D\) and an invertible matrix \(P\) such that \(A = PDP^{-1}\). (Check your answer.)

8. Consider the system
\[ x + 4y = 1 \\
\]
\[ x + y = 2 \\
\]
\[ x + y = 3. \\
\]

(a) Demonstrate that this system is inconsistent.
(b) Write down the corresponding normal equations for this system.
(c) Find the best solution(s) in the sense of least squares.
(d) Compute the least squares error for the solutions found in part (c).

9. Let \( C_k \) denote the number of cats in central park at the end of \( k \) months, and \( M_k \) the corresponding number of mice. The populations of cats and mice have been determined to evolve according to the equations

\[
C_{k+1} = .2C_k + .9M_k \\
M_{k+1} = -.6C_k + 2.3M_k.
\]

It is known that the eigenvalues of the matrix

\[
\begin{bmatrix}
.2 & .9 \\
-.6 & 2.3
\end{bmatrix}
\]

are 2 and 0.5, with corresponding eigenvectors \((1, 2)\) and \((3, 1)\), respectively.

(a) Suppose that initially there are 20 cats and 100 mice. Find a formula, in terms of \( k \), for the number of cats \( C_k \) after \( k \) months.
(b) Some time later a survey finds that there are 200,000 cats in the park. Approximately how many cats will there be one month later? Justify.

10. True or False. (If true, justify; if false, provide a counterexample and explanation.)
(a) Every \( 2 \times 2 \) matrix can be diagonalized.
(b) If \( A \) and \( B \) have the same eigenvalues, then \( A \) and \( B \) are similar.
(c) If \( A^2 = 0 \), then \( A = 0 \).
(d) If \( A \) is a \( 5 \times 7 \) matrix and \( \text{Nul}(A) \) has dimension three, then the system \( Ax = b \) will be inconsistent for certain vectors \( b \).