M340L Final Exam Solutions, May 7, 2003

1. Let $A = \begin{pmatrix} 1 & 0 & -1 & 2 & 3 \\ 1 & 1 & 6 & 5 & 4 \\ 2 & 1 & 5 & 7 & 8 \\ 5 & 2 & 9 & 16 & 16 \end{pmatrix}$ and let $b = \begin{pmatrix} 10 \\ 13 \\ 25 \\ 54 \end{pmatrix}$. The augmented

matrix $[Ab]$ is row-equivalent to $\begin{pmatrix} 1 & 0 & -1 & 2 & 0 & 4 \\ 0 & 1 & 7 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

a) Find all solutions to $Ax = b$. Express your answers in parametric form.

The row-reduced equations read $x_1 = 4 + x_3 - 2x_4$, $x_2 = 1 - 7x_3 - 3x_4$, $x_5 = 2$, while x_3 and x_4 are free ($x_3 = x_3$ and $x_4 = x_4$). Thus

$$ x = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ -7 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}. $$

b) Find a basis for the column space of A.

The first, 2nd and 5th columns are pivot columns, so our basis is

$$ \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 8 \\ 16 \end{pmatrix} \right\} $$

c) Find a basis for the null space of A.

This is the same as part (a), only with zero on the right hand side. Our

basis is $\begin{pmatrix} 1 \\ -7 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

d) Find a basis for $(Col(A^T))^{\perp}$.

This is the same thing as the null space of A, so the answer is the same as (c).

2. For each of these matrices, (i) find the determinant, (ii) state whether the matrix is invertible, and (iii) either find the inverse of the matrix (if it is invertible) or find a nonzero solution to $Ax = 0$ (if it isn't).

a) $\begin{pmatrix} 9 & 6 \\ 12 & 8 \end{pmatrix}$

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The determinant is 0, the matrix is NOT invertible, and a nontrivial solution to $Ax = 0$ is $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

b) $\begin{pmatrix} 1 & 2 & -3 & 9 \\ 0 & 1 & 3 & 0 \\ 0 & 3 & 10 & 0 \\ 0 & 3 & 6 & -1 \end{pmatrix}$.

The determinant is -1 (expand about the first column, and then about the last column), the matrix is invertible, and the inverse (obtained by row-reducting $[AI]$) is $\begin{pmatrix} 1 & -137 & 36 & 9 \\ 0 & 10 & -3 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 12 & -3 & -1 \end{pmatrix}$.

3. In the space P_2 of quadratic polynomials, let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis, and let $\mathcal{B} = \{1 + t^2, 2t + t^2, 1 + 3t + 2t^2\}$ be an alternate basis

a) Find the change-of-basis matrix P_{EB} that converts from coordinates in the B basis to coordinates in the E basis.

$$P_{EB} = ([b_1]_E [b_2]_E [b_3]_E) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$$

b) Find the change-of-basis matrix P_{BE} that converts from coordinates in the E basis to coordinates in the B basis.

$$P_{BE} = (P_{EB})^{-1} = \begin{pmatrix} -1 & -1 & 2 \\ -3 & -1 & 3 \\ 2 & 1 & -2 \end{pmatrix} \text{ (computed by row-reducing } [P_{EB}I].)$$

c) Compute the coordinates, in the B basis, of the following four vectors: $3 + 4t - t^2, 5 - t + 2t^2, 1, 1 + 3t + 2t^2$

Just multiply P_{BE} by the coordinates of these vectors in the E basis to get: $\begin{pmatrix} -9 \\ -16 \\ 12 \end{pmatrix}, \begin{pmatrix} 0 \\ -8 \\ 5 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

4. Let $T : P_2 \to \mathbf{R}^4$ be defined by $T(p) = \begin{pmatrix} p(1) \\ p'(1) \\ p(2) \\ p'(2) \end{pmatrix}$, where p' is the derivative of p with respect to t. Find the matrix of this linear transformation relative to the standard basis.

Since $T(1) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $T(t) = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$ and $T(t^2) = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 4 \end{pmatrix}$, the matrix is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \\ 0 & 1 & 4 \end{pmatrix}.$$

5. a) Find a 3×3 matrix A whose eigenvalues are 1, 0 and -1, and whose

corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$.

$A = PDP^{-1}$ where $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Comput-

ing $P^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}$ and multiplying gives $A = \begin{pmatrix} 1 & 1 & = 1 \\ 0 & 3 & -2 \\ -2 & 7 & -4 \end{pmatrix}$.

b) Compute A^{37}.

$A^{37} = PD^{37}P^{-1}$. But D^{37} is easily seen to equal D, so $A^{37} = PDP^{-1} =$
$A = \begin{pmatrix} 1 & 1 & = 1 \\ 0 & 3 & -2 \\ -2 & 7 & -4 \end{pmatrix}$.

6. In this problem, we model the spread of an epidemic. Let $S(k)$ be the number of sick people in week k, and let $I(k)$ be the number of people who are infected, but not yet sick. Each week, a sick person will infect 6 others, while an infected person will become sick. (In our model, nobody ever recovers or dies). That is,

$$\begin{aligned} S(k+1) &= S(k) + I(k) \\ I(k+1) &= 6S(k) \end{aligned}$$

Letting $x(k) = \begin{pmatrix} S(k) \\ I(k) \end{pmatrix}$, this boils down to $x(k+1) = \begin{pmatrix} 1 & 1 \\ 6 & 0 \end{pmatrix} x(k)$.

a) Find the eigenvalues and corresponding eigenvectors of the matrix.

Eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$ and eigenvectors are $b_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and

$b_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ (or any nonzero multiple of these choices).

b) In the long run, what will be the ratio of sick to infected (but not yet sick) people?

In the long run, the coefficient of b_1 dominates, so the ratio of sick to infected approaches that of b_1, namely 1:2.

c) If there are 3 sick people and 1 infected person in week zero, how many sick and infected people will there be in week k?

$$x(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 2b_1 + b_2, \text{ so } x(k) = 2(3^k)b_1 + (-2)^k b_2 = \begin{pmatrix} 2 \cdot 3^k + (-2)^k \\ 4 \cdot 3^k - 3(-2)^k \end{pmatrix}.$$

7. Let $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$, and $b = \begin{pmatrix} 1 \\ 2 \\ 4 \\ -3 \end{pmatrix}$. Let $V = Span\{v_1, v_2\}$.

a) Compute $P_V b$.

Since v_1 and v_2 are orthogonal, $P_V b = P_{v_1} b + P_{b_2} v = \frac{v_1 \cdot b}{v_1 \cdot v_1} v_1 + \frac{v_2 \cdot b}{v_2 \cdot v_2} v_2 =$

$\frac{4}{4} v_1 - \frac{6}{6} v_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}$.

b) Find the distance from b to the plane V.

The distance is $|b - P_V b| = |(1, -1, 3, -3)^T| = \sqrt{20} = 2\sqrt{5}$.

c) Find a least-squares solution to $\begin{pmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} x = b$.

We have already seen that the projection of b is $1v_1 + (-1)v_2$, so our least-squares solution is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. You can also get this answer by solving $A^T A x = A^T b$.

8. a) Find an orthogonal basis for the column space of $\begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 5 \\ 1 & 1 & 0 \\ 0 & -1 & 3 \end{pmatrix}$.

$$v_1 = x_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 = x_3 - 3v_1 - 2v_2 = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 5 \end{pmatrix}$$

b) Find the projection of $\begin{pmatrix} 3 \\ 2 \\ 3 \\ 6 \end{pmatrix}$ onto this space.

The vectors v_1, v_2, v_3 give an orthogonal basis for this column space, so $P_{col(A)} = P_{v_1} + P_{v_2} + P_{v_3}$, and $P_{Col(A)}b = \frac{b \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{b \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{b \cdot v_3}{v_3 \cdot v_3} v_3 = 3v_1 - v_2 + v_3 = \begin{pmatrix} 3 \\ 2 \\ 3 \\ 6 \end{pmatrix}$. The vector b was already IN the column space, so its projection is itself.

9. True-False. Indicate whether each of these statements is true or false. If a statement is sometimes true and sometimes false, write "false". You do NOT have to justify your answers. There is no penalty for wrong answers, so go ahead and guess if you are unsure of your answer.

a) The equation $Ax = b$ has a solution if, and only if, b is in the span of the columns of A.

True.

b) The equation $Ax = b$ has a solution if, and only if, the augmented matrix $[Ab]$ has a pivot position in each row.

False. It has a solution if there is NOT a pivot in the last COLUMN.

c) If the $m \times n$ matrix A has a pivot in each column, then the columns of A are linearly independent.

True.

d) If the $m \times n$ matrix A has a pivot in each column, then the columns of A spen $\mathbf{R}^m$.

False. Having a pivot in each column implies the columns are linearly independent. Having a pivot in each ROW implies that they span $\mathbf{R}^m$.

e) Every linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$ can be represented by an $m \times n$ matrix.

True.

f) If A, B and C are matrices such that the product ABC makes sense, then $(ABC)^T = C^T B^T A^T$.

True.

g) If the determinant of a square matrix A is zero, then A is invertible.

False. If the determinant is zero, the matrix is NOT invertible.

h) Given vectors $v_1, \ldots, v_p \in \mathbf{R}^n$, the set of all linear combinations of these vectors is a subspace of $\mathbf{R}^n$.

True. The span of several vectors is always a subspace.

i) If two matrices are row-equivalent, then their column spaces have the same dimension.

True.

j) The row space of a matrix has the same dimension as the column space.

True. Both dimensions equal the rank of the matrix.

k) $\mathbf{R}^2$ is a subspace of $\mathbf{R}^3$.

False. $\mathbf{R}^2$ does not sit inside $\mathbf{R}^3$.

l) The range of $T(x) = Ax$ is the same as the column space of A.

True.

m) If $H = Span\{b_1, \ldots, b_p\}$, then $\{b_1, \ldots, b_p\}$ is a basis for H.

False. They may not be linearly independent.

n) The dimension of the null space of a matrix is the number of free variables in the equation $Ax = 0$.

True.

o) If A is a 5×3 matrix, then the null space of A is at least 2-dimensional.

False. This would be true for a 3×5 matrix, not a 5×3.

p) A number c is an eigenvalue of A if and only if $det(A - cI) = 0$.

True.

q) If the characteristic polynomial of a 4×4 matrix A is $p_A(\lambda) = \lambda(\lambda - 1)(\lambda + 3)(\lambda - 15)$, then A is diagonalizable.

True. The eigenvalues are all distinct.

r) Every 2×2 matrix has at least one eigenvalue, but it may be complex.

True.

s) If $A = PDP^{-1}$, with D diagonal, then each column of P is an eigenvector of A.

True.

t) If $Ax = 0$, then x is in $(Col(A))^{\perp}$.

False. x is in $(Col(A^T))^{\perp}$.

u) If two nonzero vectors are orthogonal, they are linearly independent.

True.

v) If two nonzero vectors are linearly independent, they are orthogonal.

False. (e.g., $v = (1, 0)^T$ and $w = (1, 1)^T$.)

w) If $y = z_1 + z_2$, where $z_1 \in W$ and $z_2 \in W^{\perp}$, then $P_W y = z_1$.

True.

x) The equation $Ax = b$ always has a least-squares solution, no matter what A and b are.

True. Least-squares solutions always exist.

y) If $b \in Col(A)$, then $Ax = b$ has a unique least-squares solution.

False. Uniqueness has to do with the columns of A being linearly independent.