Lie Groups Solutions, Problem Set # 4

Section 2.5:

2: If $F$ is $g$–stable, then $Xv \in F$ for all $X \in g$, $v \in F$. Likewise, $X^2v = X(Xv) \in F$, and by induction $X^n v \in F$. Since $F$ is a vector space, $\exp(X)v = \sum X^n v/n! \in F$, so $\exp(g)$ sends $F$ to itself. Thus the group generated by $\exp(g)$ sends $F$ to itself. Since $G$ is connected, that is all of $G$.

Conversely, if $F$ is $G$-stable and $X \in g$, then $\exp(tX)v \in F$ for all $v \in F$. Taking a derivative with respect to $t$ at $t = 0$ means that $Xv \in F$.

5: $SO(3)$ and $SU(2)$ are NOT complex, nor are $O(3)$ or $SL(2,R)$ or the Euclidean group acting on $R^2$. (Any complex group must have an even real dimension, so these 3-dimensional examples are easily eliminated). However, $SL(2,C)$ is complex, as is $SL(n,C)$, and as is $GL(n, C)$. The triangle groups of Example 6 are complex (if $E$ is a complex vector space) as is the group of affine transformations when $E$ is complex. Finally, the direct product of two complex groups is complex.

7: (a) Any path through the origin in $G$ can be written uniquely as the product of a path in $M$ and a path in $N$: $\gamma(t) = \alpha(t)\beta(t)$, and at $t = 0$ we have $d\gamma/dt = d\alpha/dt + d\beta/dt$. Thus $g = m + n$. Since $M \cap N = 1$, $m \cap n = 0$, so $g = m \oplus n$. Since $M$ is a subgroup, $m$ is a sub-algebra. Since $N$ is a normal subgroup, $n$ is an ideal.

(b) By Baker-Campbell-Haussdorff, $\exp(-X)\exp(X+Y) = \exp(Z)$, with $Z$ given by a sum of brackets. Since $n$ is an ideal, all terms of the brackets are in $n$, so $Z \in n$, and we can define $A(X)Y = Z$. Note the expression $\exp(A(X)Y)$ should be read as $\exp(A(X)Y)$, and not as $(\exp A(X))Y$.

(c) First note that, by Dynkin’s formula, $\exp(X)\exp(tY) = \exp(W(t))$, where $W(t) - X \in n$, by the same argument as above. When $N$ is Abelian, we re-do the derivation of Dynkin’s formula as follows: Let $\exp(W(t)) = \exp(X)\exp(tY)$. Then $dW/dt = e^{WY}$. However, $dW/dt = e^{W[(1 - \exp(-adW))/ad(W)]dW/dt}$, so $dW/dt = [(1 - \exp(-adW))/ad(W)]^{-1}Y$. However, acting on $n$, $ad(W) = ad(X)$, since $n$ is Abelian. Thus $dW/dt = A(X)^{-1}Y$, so $W(1) = W(0) + A(X)^{-1}Y = X + A(X)^{-1}Y$. That is, we have proven that $\exp(X)\exp(Y) = \exp(X + A(X)^{-1}Y)$. Now, replacing $Y$ with $A(X)Y$, we get $\exp(X)\exp(A(X)Y) = \exp(X + Y)$.

(d) The affine group is the set of all matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a$ invertible and $b \in E$. This is (uniquely) factored as $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{a}b \\ 0 & 1 \end{pmatrix}$.

11: Recall that we have an inner product on $g$, namely $\langle X|Y \rangle = \text{Tr}(X^*Y)$. Relative to this inner product, $(ad(Z))^* = ad(Z^*)$, where on both sides the superscript * means
adjoint. This is easily checked: \( \langle X|ad(Z)Y \rangle = Tr(X^*(ZY - YZ)) = Tr((X^*Z - ZX^*)Y) = \langle ad(Z^*)X|Y \rangle \).

(a) Since \( Z \) is self-adjoint, \( ad(Z) \) is self adjoint, hence diagonalizable with real eigenvalues, so \( g \) is the direct sum of eigenspaces with real eigenvalues.

(b) If \( X \in g_\lambda \) and \( Y \in g_\mu \), then by Jacobi, \( [Z, [X,Y]] = [[Z,X],Y] + [X,[Z,Y]] = [\lambda X,Y] + [X,\mu Y] = (\lambda + \mu)[X,Y] \).

(c) If \( X \in g_\lambda \), then \( [Z,X^*] = ZX^* - X^*Z = (-Z^*X + XZ^*)^* = -[Z^*,X]^* = -[Z,X]^* = -\lambda X^* \).

(d) The fact that \( q \) is a subalgebra follows from (b). The fact that \( k \) is a subalgebra comes from the fact that \( [X,Y]^* = -[X^*,Y^*] \). To see that \( k + q = g \) (not necessarily direct sum!), we decompose an arbitrary element of \( g \) into a \( k \) piece and a \( q \) piece. By (a), we can assume with loss of generality that \( X \in g_\lambda \). If \( \lambda \geq 0 \), then \( X \in q \). If \( \lambda < 0 \), then \( X = (X - X^*) + X^* \), with \( X - X^* \in k \) and \( X^* \in q \).

(e) (This is closely related to polar decomposition.) To see that \( L(K) = k \), note that the derivative of the equation \( k(t)^*k(t) = 1 \) at \( t = 0 \) is \( X^* + X = 0 \), where \( X = dk/dt \). Thus all elements of \( L(K) \) are anti-hermitian. Likewise, the exponential of any anti-hermitian elements of \( g \) are both unitary and in \( G \), hence in \( K \). It’s obvious that \( \exp(q) \subset N_G(q) \), and hence that \( q \subset L(Q) \). Conversely, if \( Y \in L(Q) \), then \( \exp(Yt) \in Q \), so \( \exp(Yt)X \exp(-Yt) \in q \) for all \( X \in q \), so \([Y,X] \in q \). But \( Z \in q \), so \([Y,Z] \in q \). But this means that \( Y \in q \). To show that \( G = KQ \), it suffices by problem 10 to show that \( KQ \) is closed. So suppose that we have a sequence \( k_jq_j \) that converges (in \( G \)). Since \( K \) is compact, there is a subsequence such that \( k_j \) converges. But if \( k_j \) and \( k_jq_j \) both converge, then so does \( k_j^*k_jq_j = q_j \), and we have that \( \lim k_jq_j = \lim k_j \lim q_j \in KQ \).

12: \( K = SO(2) \), and \( Q \) is the group of upper-triangular matrices with determinant 1. (Called “B” in Lemma 3B of section 2.1).

Section 2.6:

For these problems, note that the adjoint action of a group on its Lie algebra preserves a bilinear form on the Lie algebra, namely \( \langle X|Y \rangle = -Tr(XY) \). Call this form \( K \) (for Killing). The adjoint action \( Ad \) gives a homomorphism from \( G \) to \( Aut(K) \). In each example it is easy to see that the infinitesimal action \( ad \) is 1–1. Since the groups have the same dimension (in these examples), this induces a covering.

6: Define an action of \( SL(2,C) \) on \( C^3 \) as follows. First identify \( C^3 \) with the Lie algebra \( sl(2,C) \), and then take the adjoint action of \( SL(2,C) \) on \( sl(2,C) \). That is, if \( a \in SL(2,C) \) and \( X \in sl(2,C) \), let \( \rho(a)X = Ad(a)X = aXa^{-1} \). This is just the complexification of the adjoint action of \( SU(2) \) on \( su(2) \), hence is the complexification
of the action of \( SO(3) \) on \( R^3 \), hence is an action of \( SO(3, C) \) on \( C^3 \).

7: (a) The bilinear form has signature \((2,1)\), so the adjoint action gives a map \( SL(2, R) \rightarrow SO(2, 1) \). Since \( SL(2, R) \) is connected, the image is connected, hence is in the identity component of \( SO(2, 1) \). Since it is 3-dimensional, it IS the identity component.

(b) The Lie algebra is spanned by \( \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \), for which the Killing form has signature \((1,2)\). So the image of the double cover map is a connected 3-dimensional subgroup of \( SO(1, 2) = SO(2, 1) \), hence is the identity component.

(c) The Lie algebra of \( SL(2, R) \) is spanned by the anti-hermitian matrix \( X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and the Hermitian matrices \( X_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and \( X_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), while \( su(1,1) \) is spanned by the anti-hermitian matrix \( Y_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \) and the Hermitian matrices \( Y_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( Y_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \). These are conjugate by \( P = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \). That is, \( Y_k = PX_kP^{-1} \). This exponentiates to give \( SU(1, 1) = PSL(2, R)P^{-1} \).

8: (a) \( SL(2, C) \times SL(2, C) \) acts on \( M_2(C) = C^4 \) (NOT \( C^2 \) – that’s a typo) by \( X \rightarrow aXb^{-1} \). Since \( \text{det}(a) = \text{det}(b) = 1 \), this preserves the determinant of \( X \), which is a nondegenerate bilinear form on \( M_2(C) \). Hence we have a homomorphism \( SL(2, C) \times SL(2, C) \rightarrow SO(4, C) \). The groups have the same dimension, and the kernel of \( Lf \) is empty, and \( SO(4, C) \) is connected, so this is a covering map. The kernel is \( \{(1, 1), (-1, -1)\} \), so it’s a double cover.

(b) Let \( SL(2, C) \) act on the hermitian \( 2 \times 2 \) matrices (which are isomorphic to \( R^4 \), not to \( R^3 \)) by \( X \rightarrow aXa^* \). As before, this preserves the determinant, which is a bilinear form. This bilinear form has signature \((3,1)\), and the rest of the argument is as in (a).