Lie Groups Solutions, Problem Set # 5

Section 3.1:

1: (a) Given a real vector space $E$, let $F = E \otimes \mathbb{C}$, and let $C$ be complex conjugation. Conversely, given a pair $(F,C)$ with $F$ of complex dimension $n$, we can view $F$ as a real vector space of real dimension $2n$. Since $C^2 = 1$, the eigenspaces of $C$ have eigenvalues $\pm 1$, and since $Ci = -iC$, multiplication by $i$ sends each eigenspace to the other. Let $E$ be the $+1$ eigenspace, so $iE$ is the $-1$ eigenspace, so $F = E \oplus iE = E \otimes \mathbb{C}$. This shows that the correspondence is a bijection.

(b) If $E$ is a right-$H$ vector space, then $E$ is also a (right) complex vector space (of twice the dimension), since the complexes are a subset of the quaternions. Let $F$ be the same set as $E$, and let $J$ be right-multiplication by $j$. Since for any complex number $\alpha$, $\alpha j = j\bar{\alpha}$, $J$ is complex anti-linear, and since $j^2 = -1$, $J^2 = -1$. Conversely, if we have a pair $(F,J)$, then we can allow the quaternion $q = \alpha + j\beta$ to act on a vector $x$ by $x(\alpha + j\beta) = x(\alpha) + (Jx)\beta$.

Note an essential difference between the two constructions. In the real case, $E$ is a subset of $F$. In the quaternionic case, $E$ is the same set as $F$.

6: Here are two solutions: (a) Work on the Lie algebra level. $sl(2,\mathbb{R})$ is spanned by the Hermitian matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the anti-Hermitian matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Meanwhile, $su(1,1)$ is spanned by two Hermitian matrices and the anti-Hermitian matrix $B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. But $A = cBc^{-1}$, where $c = \begin{pmatrix} i & -i \\ -i & 1 \end{pmatrix}$ is the matrix of eigenvectors of $A$. Likewise, the Hermitian generators of $sl(2,\mathbb{R})$ are $Ad(c)$ of the Hermitian generators of $SU(1,1)$, and by exponentiation we see that $SL(2,\mathbb{R}) = cSU(1,1)c^{-1}$. (b) Following the hint in the book, $SL(2,\mathbb{R})$, acting on $\mathbb{R}^2$, preserves the bilinear form with matrix $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. But then acting on $\mathbb{C}^2$ it preserves the sesquilinear form with matrix $iM = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. But this is a Hermitian form of signature $(1,1)$, so $SL(2,\mathbb{R}) \subset SU(iM)$. Since the groups are connected and of the same dimension, they are in fact equal. But $SU(iM)$ is conjugate to the standard $SU(1,1)$ by a change of basis that takes $iM$ to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which is precisely the matrix $c$ listed above.

11: In all cases we count degrees of freedom in the Lie algebra. (a) There are $n^2$ variables and one constraining (trace equals zero), hence $n^2 - 1$ degrees of freedom. (b)
so(n, \mathbb{C}) is the set of anti-symmetric matrices, which are determined by the upper triangular block, with \( n(n-1)/2 \) degrees of freedom. (c) As we worked out in problem 2.2.3, the Lie algebra of \( sp(n, \mathbb{C}) \) is all block matrices \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) with \( D = -A^t \) (\( m^2 \) degrees of freedom) and with \( B \) and \( C \) symmetric (\( m(m+1)/2 \) degrees of freedom each), for a total of \( 2m^2 + m \).

12: (a) \( g_0 = su(n) \) is the set of traceless anti-Hermitian matrices, \( ig_0 \) is the set of traceless Hermitians, and \( g_0 \oplus ig_0 = g \) is the set of all traceless matrices. (b) \( g_0 \) is the set of anti-symmetric real matrices, \( ig_0 \) is the set of anti-symmetric imaginary matrices, and \( g = g_0 \oplus ig_0 \) is the set of all anti-symmetric matrices. (c) As in problem 2.2.3, \( g_0 \) is the set of all real block matrices \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), with \( D = -A^t \) and with \( B \) and \( C \) symmetric, \( ig_0 \) is the set of imaginary matrices of that form, and \( g \) is the set of all complex matrices of that form.

Section 3.2:

6: (a) For \( G = SL(n, \mathbb{C}) \), every simple matrix is diagonalizable, and the matrix of eigenvectors can be chosen to have determinant 1 (just rescale one of the eigenvectors). For the other classical groups, we have a bit more work to do. Suppose that \( G \) preserves the bilinear form \( \phi \). Then for any \( a \in G \), and any eigenvectors \( v_1, v_2 \) of \( a \) with eigenvalues \( \lambda_1 \) and \( \lambda_2 \), \( \phi(v_1, v_2) = \phi(av_1, av_2) = \lambda_1 \lambda_2 \phi(v_1, v_2) \). That is, either \( \lambda_1 \lambda_2 = 1 \) or the two eigenvectors are \( \phi \)-orthogonal. However, \( \phi \) is non-degenerate, so it can’t be that EVERY eigenvector is orthogonal to \( v_j \). For each \( j \), there must be an eigenvector \( w_j \) whose eigenvalue is \( \lambda_j^{-1} \), and for which \( \phi(v_j, w_j) = 1 \). For the moment, suppose all of the eigenvalues of \( a \) are distinct, and that \( G = SO(2n, \mathbb{C}) \) or \( Sp(n, \mathbb{C}) \). Then we can choose list our eigenvectors in the form \( v_1, \ldots, v_n, w_1, \ldots, w_n \). The matrix that has these vectors as its columns will be in \( G \). (Seeing that it preserves \( \phi \) is precisely the \( \phi \)-orthonormality of the eigenvectors. Seeing that it has determinant \(+1\) is subtler.) If \( G = SO(2n+1, \mathbb{C}) \), then there is an additional eigenvector with eigenvalue 1, which goes last. Finally, if there are repeated eigenvalues, then we must do a Gram-Schmidt-like change-of-basis within each eigenspace to ensure that \( \phi(v_j, v_k) = 0 = \phi(w_j, w_k) \) and that \( \phi(v_j, w_k) = 1 \) if \( j = k \) and zero otherwise.

(b) If \( X \in g \) is semi-simple, then \( \exp(tX) \) is a semi-simple element of \( G \), so by (a) its eigenvectors can be assembled into an element of \( G \). But for \( t \) small enough, all eigenvectors of \( \exp(tX) \) are eigenvectors of \( X \), so \( X \) is conjugate to an element of \( h \) by \( G \).

(c) Pick an element \( X \in a \) such that \( X \) has a maximal number of distinct eigenvalues. By (b), \( X \) is conjugate (by \( G \)) to a diagonal matrix, and without loss of generality we can group the repeated eigenvalues together. Any matrix \( Y \in a \) commutes with
$X$, and so must be block-diagonal, with blocks corresponding to the eigenspaces of $X$. Now I claim that the blocks in $Y$ are all proportional to the identity, for otherwise, by first-order perturbation theory, for small $t$, $X + tY$ would have more distinct eigenvalues than $X$, which contradicts the maximality condition. Thus every element of $a$ is diagonal in this basis, so $a$ is conjugate to a subalgebra of $h$.

(d) Every connected abelian subgroup $A$ of $G$ consisting only of semi-simple elements is generated by its Lie algebra $a$, which is, by (c), conjugate to a subalgebra of $h$. So $A = \Gamma(a)$ is conjugate to a subgroup of $H = \Gamma(h)$.

(e) By (c), there is $g \in G$ such that $Ad(g)a \subset h$. But then $a \subset Ad(g^{-1})h$. Since $a$ is a maximal abelian subgroup, $a$ must equal $Ad(g^{-1})h$, so $a$ is conjugate to $h$.

(f) Since $A$ is connected, $A = \Gamma(L(A))$. Note that $L(A)$ is a maximal abelian subgroup consisting of semi-simple elements, so by (e), $L(A)$ is conjugate to $h$. But then $A = \Gamma(L(A))$ is conjugate to $H = \Gamma(h)$.

(g) Take $G = SO(E)$, and consider $A$ to be the diagonal matrices with entries $\pm 1$, relative to the basis of problem 1. (This is equivalent to changing the bilinear form to the one represented by the identity matrix). These are the only diagonal matrices in $G$. The only matrices that commute with all of $A$ are diagonal matrices, hence $A$ is a maximal abelian subgroup consisting of semi-simple elements. However $A$, being finite, is not conjugate to $H$.

(h) We already did this in class. The group of $2n \times 2n$ matrices with block form

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

is maximal abelian of dimension $n^2$, but is not conjugate to $H$ (which has dimension $2n$).