We aim to show that the contrapositive of a statement is equivalent to the original statement. In other words, that “P implies Q” is the same as “(not Q) implies (not P)”. First we need some definitions and notation.

Let $A$ be our universe. Let $S_P$ denote the set of events in which $P$ is true. We’ll denote “not $P$” by $\sim P$, so $S_{\sim P} = A - S_P$ is the set of events in which $P$ is false. Note that $A = S_P \cup S_{\sim P} = S_Q \cup S_{\sim Q}$. The phrase “$P$ implies $Q$” is defined to mean $S_P \subseteq S_Q$. The empty set is denoted $\emptyset$.

Now for a lemma: “$P$ implies $Q$” is equivalent to $S_P \cap S_{\sim Q} = \emptyset$.

The lemma is proved in two steps. First we show that $S_P \cap S_{\sim Q} = \emptyset$ implies $S_P \subseteq S_Q$. The second term is disjoint from $S_Q$, so the only way it can be a subset of $S_Q$ is if it is empty. Thus $S_P \cap S_{\sim Q} = \emptyset$.

Step 2: Suppose $S_P \cap S_{\sim Q} = \emptyset$. As before, we have that $S_P = (S_P \cap S_Q) \cup (S_P \cap S_{\sim Q})$. The first term is a subset of $S_Q$ and the second term is empty, so $S_P \subseteq S_Q$. That completes the proof of the lemma.

Once we have the lemma, we note that “$P$ implies $Q$” means $S_P \cap S_{\sim Q} = \emptyset$, while “(not $Q$) implies (not $P$)”, by the same lemma, means $S_{\sim Q} \cap S_{\sim P} = \emptyset$. But $\sim \sim P = P$, so the conditions are the same.

That completes the formal proof. Let me restate this in informal terms. “$P$ implies $Q$” means that $Q$ is true whenever $P$ is true. In other words, that you can’t have $P$ true and $Q$ false. That’s the lemma. Then, “(not $Q$) implies (not $P$)” means that you can’t have (not $Q$) true and (not $P$) false. But that’s the same thing!