M367K Final Exam Solutions, May 10, 2008

1. Let X and Y be sets, and let $f : X \to Y$ be a function. (a) Show that for any $A \subset X$, and $B \subset Y$, $A \subset f^{-1}(f(A))$ and $f(f^{-1}(B)) \subset B$. (b) State (with proof!) an additional assumption on f that implies that $A = f^{-1}(f(A))$. (c) State (with proof!) an additional assumption on f that implies that $B = f(f^{-1}(B))$.

(a) If $x \in A$, then $f(x) \in f(A)$, so $x \in f^{-1}(f(A))$. $C = f^{-1}(B)$ consists of all elements that map into B, so $f(f^{-1}(B)) = f(C)$ lies in B.

(b) Claim: If f is 1-1, then $A = f^{-1}(f(A))$. Proof: Every element of $f(A)$ is hit by at most one element of X. However, it is (by definition!) hit by an element of A, so it cannot be hit by anything outside of A. In other words, $f^{-1}(f(A)) \subset A$. Since we have already shown that $A \subset f^{-1}(f(A))$, $A = f^{-1}(f(A))$.

(c) Claim: If f is onto, then $B = f(f^{-1}(B))$. Proof: If f is onto, then every point in B is hit by at least one point in X. By definition, such a point is in $f^{-1}(B)$, so the image of $f^{-1}(B)$ contains all of B. But we already showed it was a subset of B, so it must equal B.

2. Let n and m be positive integers with $n > m$. Show that there is no 1-1 map from $\{1, \dots, n\}$ to $\{1, \dots, m\}$.

[In other words, prove the pigeonhole principle. Unfortunately, most people *assumed* the pigeonhole principle, or facts about cardinality that are based on the pigeonhole principle. That's circular reasoning!]

We prove this by induction on m. Let $f : S_n \to S_m$ be a map, where $S_n = \{1, \dots, n\}$. If $m = 1$, then all points map to 1, so f is not 1-1. If $m > 1$, then there are three cases. If m is not in the image, then f maps S_n to S_{m-1}. By the inductive hypothesis, f is not 1-1. If $f^{-1}(m)$ consists of more than one point, then f is not 1-1. If $f^{-1}(m)$ consists of a single point x, then f maps $S_n - \{x\}$ to S_{m-1}. But the points of $S_n - \{x\}$ are in 1-1 correspondence with the points of S_{n-1}. Since there are no 1-1 maps from S_{n-1} to S_{m-1} (by the inductive hypothesis), f restricted to $S_n - \{x\}$ cannot be 1-1, and so f cannot be 1-1.

3. (a) Consider the following functions from R to R^{Z_+}. For each, and for each natural topology on R^{Z_+} (product, uniform, box), indicate (with justification!) whether the function is continuous.

$$f(t) = (\sin(t), \sin(2t), \sin(3t), \dots)$$

$$g(t) = (\sin(t), \sin(t), \sin(t), \ldots)$$
$$h(t) = (\sin(t), \sin(t/2), \sin(t/3), \ldots)$$

In the product topology, a map into R^{Z_+} is continuous if and only if each of its component functions is continuous. Since $\sin(nt)$ and $\sin(t/n)$ are continuous, all three functions are continuous in the product topology.

In the uniform topology, we consider the preimage of a ball of size ϵ around an arbitrary image point $f(t)$ (or $g(t)$ or $h(t)$). g^{-1} of this ball and h^{-1} of this ball contain the interval $(t-\epsilon, t+\epsilon)$ (since $|\sin(x) - \sin(y)| \leq |x - y|$, which in turn comes from the fact that the derivative of sin is bounded by 1), so g and h are continuous. But f isn't. The only point near 0 whose image lies within ϵ of $(0, 0, 0, \ldots)$ is zero itself.

In the box topology, none of these functions are continuous. Just consider the preimage of the box $(-1, 1) \times (-1/10, 1/10) \times (-1/100, 1/100) \times \cdots$. 0 is an isolated point of this preimage, so the preimage is not open.

(b) Now let $F : R^{Z_+} \to R^{Z_+}$ be given by $F(x_1, x_2, \ldots) = (x_1, 2x_2, 3x_3, \ldots)$. Is this continuous in each of the three topologies? (We'll always take the topology of the target space to be the same as the source, so there are 3 cases to check, not 9).

In the product topology and the box topology, the map is continuous, as the preimage of a basic open set is a basic open set. However, in the uniform topology the map is not continuous: the preimage of the open set $(-0.1, 0.1) \times (-0.1, 0.1) \times (-0.1, 0.1) \times \cdots$ is not open.

[Note that you cannot infer anything about F by saying that one topology is finer than another. Being finer in the target space makes it harder to be continuous, but being finer in the source space makes it easier. Being finer in both, or coarser in both, implies nothing.]

4. (a) Is the product of two path-connected spaces path-connected? (b) Is the continuous image of a path-connected space path-connected?

(a) Yes. If X and Y is path-connected, and (x_1, y_1) and (x_2, y_2) are point of $X \times Y$, then there are paths $\gamma_1 : [0, 1] \to X$, $\gamma_2 : [0, 1] \to Y$ with $\gamma_1(0) = x_1$, $\gamma_1(1) = x_2$, $\gamma_2(0) = y_1$, and $\gamma_2(1) = y_2$. Then $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is a continuous path from (x_1, y_1) to (x_2, y_2).

b) Yes. If X is path connected and $f : X \to Y$ is continuous, let y_1 and y_2 be arbitrary points of $f(X)$. Then there are points x_i with $f(x_i) = y_i$. If γ is a path from x_1 to y_1, then $f \circ \gamma$ is a path from y_1 to y_2.

5. Let X be a linearly ordered set with the order topology. Show that X is regular.

[This problem was a disaster, with most people confusing "linearly ordered set" with "subset of R" or "continuum". Immediate predecessors and successors sometimes exist, and LUBs sometimes don't! People also confused closed sets with finite unions of closed intervals, or just with closed intervals.]

Given a point $x \in X$ and an open neighborhood U of x, we will show that there exists another open neighborhood V with $\bar{V} \subset U$.

Suppose that x is neither the greatest element of X or the smallest. (We'll handle those cases later). Then there are points a and d with $a < x < d$ such that $(a, d) \subset U$. If a is not the immediate predecessor to x and d is not the immediate successor, then there are points b, c with $a < b < x < c < d$, and we can take $V = (b, c)$. If a is the immediate predecessor to x, take $V = (a, c)$, whose closure does *not* contain a. Likewise, if d is the immediate successor of x, take $V = (b, d)$, and if a, x, b are consecutive numbers take $V = (a, d) = \{x\}$, which is both open and closed.

Finally, if x is the greatest element of X, then take $V = (b, x]$ or $(a, x]$, as before, and if x is the smallest element take $V = [x, c)$ or $[x, d)$.

6. The Urysohn theorem says that a *regular* space with a countable basis is metrizable. Give an example of a *Haussdorff* space with a countable basis that isn't metrizable. (Yes, you must show that it isn't metrizable)

This boils down to finding a Haussdorff space with a countable basis that isn't regular, because if it were metrizable it would be normal (and hence is regular). There are many answers, of course, but I was thinking of R_K. If has a countable basis consisting of open intervals with rational endpoints, and these open intervals intersected with the complement of K. We have already seen that R_K isn't regular, since you can't separate the closed set K from the point 0.