M367K Final Exam, May 10, 2008

1. Let X and Y be sets, and let $f : X \to Y$ be a function. (a) Show that for any $A \subset X$, and $B \subset Y$, $A \subset f^{-1}(f(A))$ and $f(f^{-1}(B)) \subset B$. (b) State (with proof!) an additional assumption on f that implies that $A = f^{-1}(f(A))$. (c) State (with proof!) an additional assumption on f that implies that $B = f(f^{-1}(B))$.

2. Let n and m be positive integers with $n > m$. Show that there is no 1-1 map from $\{1, \ldots, n\}$ to $\{1, \ldots, m\}$.

[In other words, prove the pigeonhole principle.]

3. (a) Consider the following functions from R to R^{Z_+}. For each, and for each natural topology on R^{Z_+} (product, uniform, box), indicate (with justification!) whether the function is continuous.

$$f(t) = (\sin(t), \sin(2t), \sin(3t), \ldots)$$
$$g(t) = (\sin(t), \sin(t), \sin(t), \ldots)$$
$$h(t) = (\sin(t), \sin(t/2), \sin(t/3), \ldots)$$

(b) Now let $F : R^{Z_+} \to R^{Z_+}$ be given by $F(x_1, x_2, \ldots) = (x_1, 2x_2, 3x_3, \ldots)$. Is this continuous in each of the three topologies? (We'll always take the topology of the target space to be the same as the source, so there are 3 cases to check, not 9).

4. (a) Is the product of two path-connected spaces path-connected? (b) Is the continuous image of a path-connected space path-connected?

5. Let X be a linearly ordered set with the order topology. Show that X is regular.

6. The Urysohn theorem says that a *regular* space with a countable basis is metrizable. Give an example of a *Haussdorff* space with a countable basis that isn't metrizable. (Yes, you must show that it isn't metrizable)