M367K First Midterm Exam, February 7, 2008

1. Prove that every positive integer can be written as a sum of distinct powers of 2. (For instance, $13 = 1 + 4 + 8$.)

This is done using strong induction. It's certainly true for $n = 1$ and $n = 2$. Now suppose it's true for n up through $k - 1$. We need to prove that k can be written as a sum of distinct powers of 2.

If k is itself a power of 2, there is nothing to prove. Otherwise, there is an integer m such that $2^m < k < 2^{m+1}$. Let $s = k - 2^m$. Since $s < 2^m < k$, s can be written as a sum of distinct powers of 2, and none of these powers is 2^m. (Since all of the powers of 2 are positive, any sum involving 2^m must be at least 2^m.) Adding 2^m to the sum for s give a sum for k that uses each power of 2 at most once.

Note that this proof involved strong induction, not regular induction. There are many ways to prove this theorem, but as far as I know they all involve strong induction.

2. A real number is *algebraic* if it is a root of a polynomial with integer coefficients. For instance, $5/3$ is a root of $3x - 5$ and $\sqrt{2}$ is a root of $x^2 - 2$. Show that the set of algebraic real numbers is countable. [Hint: First show that the set of polynomials with integer coefficients in countable.]

Recall that the countable union of countable sets is countable (the zig-zag argument) and that the finite product of countable sets is countable (the zig-zag argument plus induction on the number of terms in the product).

The set of all n-th order polynomials with integer coefficients is just Z^{n+1}, which is the finite product of countable sets, hence is countable. The set of all polynomials with integer coefficients is the union over n of the set of n-th order polynomials. This is a countable union of countable sets, hence is countable. Each polynomial has a finite (and therefore countable) set of roots, so the set of algebraic numbers is the countable union of finite (and therefore countable) sets, and so is countable.

3. Consider the set $L = \{a, b, \ldots, z\}^{Z_+}$ of all infinite words in the roman alphabet. Give this set the dictionary order. Answer the following questions about L. In all cases, you **must** explain your answers. A little handwaving is OK — there isn't enough time to be completely rigorous about everything — but your reasoning should be clear.

a) Which points have immediate successors? Which have immediate predecessors?

The only points with immediate successors are the words ending in an infinite string of z's (except for the word $zzz\ldots$, which is the largest element of L and has no successor), and the only points with immediate predecessors are those ending in an infinite string of a's (except for the word $aaa\ldots$, which is the smallest element of L and has no predecessor).

For a formal proof of this, suppose that W_1 and W_2 are infinite words with $W_1 < W_2$. Suppose that W_1 and W_2 agree in the first $n-1$ places ($n \geq 1$) and disagree in the nth place. That is, $W_1 = P\ell_1 S_1$ and $W_2 = P\ell_2 S_2$, where P is a finite word (P for prefix), ℓ_1 and ℓ_2 are letters with $\ell_1 < \ell_2$, and S_1 and S_2 are infinite words (S for suffix). If S_1 is anything but an infinite string of z's, then $P\ell_1 zzz\ldots$ is strictly between W_1 and W_2. Likewise, if S_2 is anything but an infinite string of as, then $P\ell_2 aaa\ldots$ is strictly between W_1 and W_2. In particular, W_2 cannot be the immediate successor of W_1 and W_1 cannot be the immediate predecessor of W_2.

b) Is L well-ordered? (That is, does every non-empty subset of L have a least element?)

No. Consider the set of words that are made of one b and infinitely many a's. This set has no smallest element.

c) Does L have the least-upper-bound property?

Yes. Given a non-empty set A, you can find the least-upper-bound $W \in A$ recursively as follows. The first letter of W (call it w_1) is the greatest first letter of words in A. The second letter (call is w_2) is the greatest second letter of words in A that start with w_1. In general, w_n is the greatest n-th letter of words in A that start with $w_1 \cdots w_{n-1}$.

Notice that this procedure does not necessarily give an element of A, just a least upper bound. If A is the set of words with one y and the rest z's, then the least upper bound has all z's.