M367K Second Midterm Exam Solutions, April 8, 2008

1. Comparing topologies. On R^{Z_+}, consider the metric $d(\vec{x},\vec{y}) = \sum_n 2^{-n}\bar{d}(x_n,y_n)$, where $\bar{d}(x_n,y_n) = \min(|x_n - y_n|, 1)$ is the standard bounded metric on R. Let $\mathcal{T}$ be the metric topology generated by d and let $\mathcal{T}'$ be the product topology.

a) Show that the infinite product $U = (-0.1,0.1)\times(-0.1,0.1)\times(-0.1,0.1)\times\cdots$ is *not* open in $\mathcal{T}$. [This proves that the uniform topology is not coarser than $\mathcal{T}$.]

We must show that, for some point $\vec{p}\in U$, there is no ball centered at $\vec{p}$ that fits inside U. Suppose there was such a ball, of radius ϵ, and let N be such that $2^{-N} < \epsilon$. Then $\vec{p}' = (p_1, p_2, \ldots, p_{N-1}, p_N + 1, p_{N+1}, p_{N+2}, \ldots)$ is in the ball but is not in U, which is a contradiction.

b) Show that every ϵ-ball in the d metric is open in $\mathcal{T}'$. [These balls are not cylinder sets, but are still open.]

We must show that, for any ball of radius ϵ around $\vec{x}$ and a point $\vec{y}$ in that ball, there is a cylinder set around $\vec{y}$ that lies in the ball. Let $\epsilon' = \epsilon - d(\vec{x},\vec{y})$, and pick N such that $2^{-N} < \epsilon'/2$. Then $(y_1-\epsilon'/2, y_1+\epsilon'/2)\times(y_2-\epsilon'/2, y_2+\epsilon'/2)\times\cdots\times(y_N-\epsilon'/2, y_N+\epsilon'/2)\times R\times R\times\cdots$ is our desired cylinder set.

c) Show that the cylinder set $V = V_1\times V_2\times\cdots\times V_n\times R\times R\times\cdots$, with each V_i open in R, is open in $\mathcal{T}$. [Together, parts b and c show that $\mathcal{T} = \mathcal{T}'$.]

If $\vec{x}\in V$, then there exist intervals $(x_k-\epsilon_k, x_k+\epsilon_k)\subset V_k$ for $k=1,\ldots,n$. Let ϵ be less than $\max(2^k\epsilon_k)/10$. Then the ball of radius ϵ around $\vec{x}$ lies entirely in V.

2. For each of the following spaces X and subspaces A, indicate whether A is connected, whether A is closed, and whether A is compact. Justify your answers!!

a) $X = R$ with the finite-complement topology, and $A = Z$.

Like all infinite subsets of X, A is connected and compact but not closed. There can be no separation because each open set would hit all but finitely many points of A, so any two open sets must overlap on A. The compactness of *every* subset of X was proven in class.

b) $X = R_\ell$ (that is, the real line with the lower-limit topology) and $A = [0,1]$.

A is closed (since $(-\infty,0)\cup(1,\infty)$ is open), but not connected ($A = [0,1/2)\cup[1/2,1]$) and not compact (consider the open cover $\{1\}, [0, 1-\frac{1}{n})$, where n ranges over the positive integers.

c) $X = R^2$ and $A = \{(x,y)||xy|\leq 1\}$.

A is closed (being the preimage of $[-1,1]$ under the map $f(x,y)=xy$), and connected (being path-connected), but not compact (being unbounded).

d) $X = R^{Z_+}$ with the product topology, and $A = [0,1]^{Z_+}$. [You get full credit for just answering connectedness and closure. Compactness follows from the Tychonoff theorem, which we haven't gotten to yet. However, if you can prove compactness using the methods developed so far, that's worth extra credit.]

This is connected (being path-connected by straight lines) and closed (being the infinite intersection of the closed sets $[0,1]^n \times R^{Z_+}$) and compact. To see sequential compactness, consider an arbitrary sequence of points in A. Since $[0,1]$ is compact, there is a subsequence where the first coordinate converges. There is a subsequence of *that* where the second coordinate converges, and so on. By Cantor diagonalization (i.e., taking the first term of the first subsequence, the second term of the second subsequence, and so on), we get a subsequence that converges on each coordinate, and therefore converges in the product topology.

In Chapter 5 we will see that the arbitrary (even uncountable) product of compact sets is compact. That's a hard theorem, but the argument I just gave (which is worth 10 extra credit points) is enough to show that the countable product of compact metric spaces is compact.

3. Let $f: R \to R_\ell$ be a continuous map. Show that f is constant.

Since R is connected, the image of f must be connected. However, the only non-empty connected subsets of R_ℓ consist of single points. To see this, suppose that a and b are in $A \subset R_\ell$. Then $(-\infty, (a+b)/2) \cup [(a+b)/2, \infty)$ gives a separation of A with a and b in different non-empty pieces.