M367K Second Midterm Exam, April 8, 2008

1. Comparing topologies. On R^{Z_+}, consider the metric $d(\vec{x}, \vec{y}) = \sum_n 2^{-n}\bar{d}(x_n, y_n)$, where $\bar{d}(x_n, y_n) = \max(|x_n - y_n|, 1)$ is the standard bounded metric on R. Let $\mathcal{T}$ be the metric topology generated by d and let $\mathcal{T}'$ be the product topology.

a) Show that the infinite product $U = (-0.1, 0.1) \times (-0.1, 0.1) \times (-0.1, 0.1) \times \cdots$ is *not* open in $\mathcal{T}$. [This proves that the uniform topology is not coarser than $\mathcal{T}$.]

b) Show that every ϵ-ball in the d metric is open in $\mathcal{T}'$. [These balls are not cylinder sets, but are still open.]

c) Show that the cylinder set $V = V_1 \times V_2 \times \cdots \times V_n \times R \times R \times \cdots$, with each V_i open in R, is open in $\mathcal{T}$. [Together, parts b and c show that $\mathcal{T} = \mathcal{T}'$.]

2. For each of the following spaces X and subspaces A, indicate whether A is connected, whether A is closed, and whether A is compact. Justify your answers!!

a) $X = R$ with the finite-complement topology, and $A = Z$.

b) $X = R_\ell$ (that is, the real line with the lower-limit topology) and $A = [0, 1]$.

c) $X = R^2$ and $A = \{(x, y) | |xy| \leq 1\}$.

d) $X = R^{Z_+}$ with the product topology, and $A = [0, 1]^{Z_+}$. [You get full credit for just answering connectedness and closure. Compactness follows from the Tychonoff theorem, which we haven't gotten to yet. However, if you can prove compactness using the methods developed so far, that's worth extra credit.]

3. Let $f : R \to R_\ell$ be a continuous map. Show that f is constant.