1. a) Consider the (scalar) first-order differential equation \( \frac{dy}{dx} = -\frac{3x^2 + 3y}{2y + xe^y} \), restricted to the first quadrant \((x \geq 0, y \geq 0)\). If \( y(2) = 0 \), what is \( y(0) \)? [Hint: rewrite the differential equation in exact form]

b) Consider the differential equation \( \frac{dx}{dt} = x + t \) with \( x(0) = 0 \). Find \( x(t) \) for all \( t \). [There are several ways to do this. Any correct method will get full credit.]

2a. Find the general solution to \( y'' - 3y' + 2y = 0 \).

b) Find a particular solution to \( y'' - 3y' + 2y = e^t + e^{3t} \).

c) Find the general solution to \( y'' - 2y' + 2y = 0 \).

3. Using the methods of chapter 5, find a series solution \( y = \sum_n a_n x^n \) to \( y'' - 3y' + 2y = 0 \). More precisely,

a) Find a recursion relation expressing \( a_n \) in terms of \( a_0, \ldots, a_{n-1} \). If \( y(0) = 2 \) and \( y'(0) = 3 \), find \( y(0.1) \) to 3 decimal places. [No, you don’t need a calculator for this.]

b) Now consider the equation \( x^2 y'' - 2xy' + (2 + x)y = 0 \). For what values of \( r \) might a series solution \( y = x^r \sum a_n x^n \) (with \( a_0 \) nonzero) exist? For the larger value of \( r \), take \( a_0 = 1 \) and find \( a_1 \) and \( a_2 \).

4. a) Find the general solution to the system of ODEs \( \frac{dx_1}{dt} = 2x_1 - 2x_2, \frac{dx_2}{dt} = x_1 - x_2 \). Then find a solution with the initial conditions \( x_1(0) = 8, x_2(0) = 5 \).

b) Find the general solution to the system of ODEs \( \frac{dx_1}{dt} = 2x_1 - x_2, \frac{dx_2}{dt} = 4x_1 - 2x_2 \).

5. This problem explores how a rectifier (e.g., the AC adapter on your laptop) turns AC current into DC current. The rectifier receives a signal, takes its absolute value, and then passes it through a filter to remove high-frequency components. What’s left is close to the constant voltage that your laptop wants. Let \( f(x) = \sin(x) \) (that’s the wall voltage), and let \( g(x) = |f(x)| \). Think of both of them as periodic functions with period \( 2\pi \).

a) Compute the Fourier coefficients \( \hat{f}_n \) for all \( n \).

b) Compute the Fourier coefficients \( \hat{g}_n \) for all \( n \). [Many of these are zero by symmetry. The rest require integration.]