# M427K First Midterm Exam Solutions, February 6, 2008

1. Consider the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}, \qquad y(-1) = 3.$$

Find $y(3)$.

Since $xdx + ydy = 0$, we have $x^2 + y^2 = c$. Plugging in $y(-1) = 3$ gives $c = 10$, so $y = \sqrt{10 - x^2}$. Note that we use the positive square root, since $y(-1)$ is positive. Plugging in $x = 3$ gives $y = 1$.

2. There is a radioactive material A that breaks down to material B at a rate $r$. That is, if $x(t)$ represents the amount of material A at time $t$, then $dx/dt = -rx$. B breaks down to C, also at rate $r$. We start with 1 kg of A, no B, and no C.

a) Find $x(t)$ for all time.

This is exponential decay. $x(t) = Ce^{-rt}$. Since $x(0) = 1$, $C = 1$ and $x(t) = e^{-rt}$.

b) Let $y(t)$ be the amount of material B at time $t$. Write down a differential equation for $y$, relating $dy/dt$ to $x$ and $y$.

The rate that B is being produced is $rx$, and the rate that B is being broken down is $ry$, so $\frac{dy}{dt} = rx - ry$.

c) Plug in your answer for part (a) to part (b) to get a differential equation for $y$, relating $dy/dt$ to $y$ and $t$. The variable $x$ should no longer appear in this equation.

$\frac{dy}{dt} = re^{-rt} - ry$, or $\frac{dy}{dt} + ry = re^{-rt}$.

d) Solve this equation to get $y(t)$ as an explicit function of $t$. (Note that $y(0) = 0$.)

This is a linear equation with integrating factor $e^{rt}$. Since $\frac{d}{dt}(e^{rt}y) = r$, we have $e^{rt}y = rt + c$, so $y(t) = rte^{-rt} + ce^{-rt}$. Plugging in $y(0) = 0$ gives $c = 0$ and $y(t) = rte^{-rt}$.

3. Consider the differential equation $\frac{dy}{dx} = 2xy$ with initial condition $y(0) = 1$.

a) Rewrite this as an integral equation.

$y(x) = 1 + \int_0^x 2zy(z)dz$

b) Find three approximate solutions to this equation by Picard iteration, starting with $y_0(t) = 1$. That is, write down $y_1(t)$, $y_2(t)$ and $y_3$. (Don't worry about what interval you're working on – this procedure converges for all values of $t$.)

Sorry about the typo! That should have been $y_0(x)$, $y_1(x)$, $y_2(x)$ and $y_3(x)$, not $y_0(t)$, etc.

$y_1(x) = 1 + \int_0^x 2zy_0(z)dz = 1 + \int_0^x 2zdz = 1 + x^2.$
$y_2(x) = 1 + \int_0^x 2zy_1(z)dz = 1 + \int_0^x 1 + 2z + 2z^3dz = 1 + x^2 + x^4/2.$
$y_3(x) = 1 + \int_0^x 2zy_2(z)dz = 1 + \int_0^x 1 + 2z + 2z^3 + z^5dz = 1 + x^2 + x^4/2 + x^6/6.$

c) Solve the differential equation *exactly*, using whatever method you wish. The first few terms in the Taylor series for this exact solution should agree with the approximate solutions you found in (b).

$\frac{dy}{y} = 2xdx$, so $\ln|y| = x^2 + c$. Plugging in the initial condition gives $c = 0$ and $y > 0$, so $\ln(y) = x^2$, so $y = e^{x^2}$, whose Taylor series is $1 + x^2 + x^4/2 + x^6/3! + x^8/4! + \cdots$

4a) Consider the differential equation $\dfrac{dx}{dt} = \dfrac{1}{2}\sin\left(\dfrac{x^2}{\pi}\right)$. Find all the fixed points and indicate which are stable and which are unstable. (Warning: some fixed points may be neutral.)

This is of the form $\frac{dx}{dt} = f(x)$ with $f(x) = \frac{1}{2}\sin\left(\frac{x^2}{\pi}\right)$. The fixed points are where $\frac{x^2}{\pi}$ is a multiple of $\pi$, so $x$ is of the form $\pm\pi\sqrt{n}$ where $n$ a non-negative integer. That is $x = 0, \pm\pi, \pm\sqrt{2}\pi, \pm\sqrt{3}\pi$, etc.

$f'(x) = \frac{x}{\pi}\cos\left(\frac{x^2}{\pi}\right)$. This is positive if $x > 0$ and $n$ is even or if $x < 0$ and $n$ is odd. It's negative if $x < 0$ and $n$ is even or if $x > 0$ and $n$ is odd. It's zero if $n = 0$. Thus the stable fixed points are $\{\pi, -\sqrt{2}\pi, \sqrt{3}\pi, -2\pi, \sqrt{5}\pi, -\sqrt{6}\pi, \ldots\}$, the unstable fixed points are $\{-\pi, \sqrt{2}\pi, -\sqrt{3}\pi, 2\pi, -\sqrt{5}\pi, \sqrt{6}\pi, \ldots\}$, and 0 is a neutral fixed point.

b) Now consider the *difference* equation $x_{n+1} = x_n + \dfrac{1}{2}\sin\left(\dfrac{x_n^2}{\pi}\right)$. Find all the fixed points and indicate which are stable and which are unstable. (Same warning as before.) You do *not* have to find and classify the periodic points; just the fixed points.

The fixed points are exactly as in part (a), only now $f(x) = x + \frac{1}{2}\sin\left(\frac{x^2}{\pi}\right)$, so $f'(x) = 1 + \frac{x}{\pi}\cos\left(\frac{x^2}{\pi}\right) = 1 \pm \sqrt{n}$. This is at least 1 in magnitude unless $\sqrt{n} < 2x$ and we are subtracting. The only three stable fixed points are $\pi$, $-\sqrt{2}\pi$, and $\sqrt{3}\pi$. The fixed points $x = -2\pi$ and $x = 0$ are neutral, and all other fixed points are unstable.