1. Consider the differential equation
\[
\frac{dy}{dx} = -\frac{x}{y}, \quad y(-1) = 3.
\]
Find \(y(3)\).

Since \(xdx + ydy = 0\), we have \(x^2 + y^2 = c\). Plugging in \(y(-1) = 3\) gives \(c = 10\), so \(y = \sqrt{10 - x^2}\). Note that we use the positive square root, since \(y(-1)\) is positive. Plugging in \(x = 3\) gives \(y = 1\).

2. There is a radioactive material A that breaks down to material B at a rate \(r\). That is, if \(x(t)\) represents the amount of material A at time \(t\), then \(dx/dt = -rx\). B breaks down to C, also at rate \(r\). We start with 1 kg of A, no B, and no C.

a) Find \(x(t)\) for all time.

This is exponential decay. \(x(t) = Ce^{-rt}\). Since \(x(0) = 1\), \(C = 1\) and \(x(t) = e^{-rt}\).

b) Let \(y(t)\) be the amount of material B at time \(t\). Write down a differential equation for \(y\), relating \(dy/dt\) to \(x\) and \(y\).

The rate that B is being produced is \(rx\), and the rate that B is being broken down is \(ry\), so \(\frac{dy}{dt} = rx - ry\).

c) Plug in your answer for part (a) to part (b) to get a differential equation for \(y\), relating \(dy/dt\) to \(y\) and \(t\). The variable \(x\) should no longer appear in this equation.

\[
\frac{dy}{dt} = re^{-rt} - ry, \text{ or } \frac{dy}{dt} + ry = re^{-rt}.
\]

d) Solve this equation to get \(y(t)\) as an explicit function of \(t\). (Note that \(y(0) = 0\).)

This is a linear equation with integrating factor \(e^{rt}\). Since \(\frac{d}{dt}(e^{rt}y) = r\), we have \(e^{rt}y = rt + c\), so \(y(t) = rte^{-rt} + ce^{-rt}\). Plugging in \(y(0) = 0\) gives \(c = 0\) and \(y(t) = rte^{-rt}\).

3. Consider the differential equation \(\frac{dy}{dx} = 2xy\) with initial condition \(y(0) = 1\).

a) Rewrite this as an integral equation.

\[
y(x) = 1 + \int_0^x 2zy(z)dz
\]
b) Find three approximate solutions to this equation by Picard iteration, starting with $y_0(t) = 1$. That is, write down $y_1(t)$, $y_2(t)$ and $y_3$. (Don’t worry about what interval you’re working on – this procedure converges for all values of $t$.)

Sorry about the typo! That should have been $y_0(x)$, $y_1(x)$, $y_2(x)$ and $y_3(x)$, not $y_0(t)$, etc.

\[
y_1(x) = 1 + \int_0^x 2zy_0(z)\,dz = 1 + \int_0^x 2z\,dz = 1 + x^2.
\]

\[
y_2(x) = 1 + \int_0^x 2zy_1(z)\,dz = 1 + \int_0^x 1 + 2z + 2z^3\,dz = 1 + x^2 + x^4/2.
\]

\[
y_3(x) = 1 + \int_0^x 2zy_2(z)\,dz = 1 + \int_0^x 1 + 2z + 2z^3 + z^5\,dz = 1 + x^2 + x^4/2 + x^6/6.
\]

c) Solve the differential equation exactly, using whatever method you wish. The first few terms in the Taylor series for this exact solution should agree with the approximate solutions you found in (b).

\[
\frac{du}{y} = 2xdx, \text{ so } \ln|y| = x^2 + c. \text{ Plugging in the initial condition gives } c = 0 \text{ and } y > 0, \text{ so } \ln(y) = x^2, \text{ so } y = e^{x^2}, \text{ whose Taylor series is } 1 + x^2 + x^4/2 + x^6/3! + x^8/4! + \cdots
\]

4a) Consider the differential equation $\frac{dx}{dt} = \frac{1}{2} \sin \left( \frac{x^2}{\pi} \right)$. Find all the fixed points and indicate which are stable and which are unstable. (Warning: some fixed points may be neutral.)

This is of the form $\frac{dx}{dt} = f(x)$ with $f(x) = \frac{1}{2} \sin \left( \frac{x^2}{\pi} \right)$. The fixed points are where $\frac{x^2}{\pi}$ is a multiple of $\pi$, so $x$ is of the form $\pm \pi \sqrt{n}$ where $n$ a non-negative integer. That is $x = 0, \pm \pi, \pm \sqrt{2} \pi, \pm \sqrt{3} \pi$, etc.

$f'(x) = \frac{x}{\pi} \cos \left( \frac{x^2}{\pi} \right)$. This is positive if $x > 0$ and $n$ is even or if $x < 0$ and $n$ is odd. It’s negative if $x < 0$ and $n$ is even or if $x > 0$ and $n$ is odd. It’s zero if $n = 0$. Thus the stable fixed points are $\{\pi, -\sqrt{2} \pi, \sqrt{3} \pi, -2 \pi, \sqrt{5} \pi, -\sqrt{6} \pi, \ldots\}$, the unstable fixed points are $\{-\pi, \sqrt{2} \pi, -\sqrt{3} \pi, 2 \pi, -\sqrt{5} \pi, \sqrt{6} \pi, \ldots\}$, and 0 is a neutral fixed point.
b) Now consider the difference equation $x_{n+1} = x_n + \frac{1}{2} \sin \left( \frac{x_n^2}{\pi} \right)$. Find all the fixed points and indicate which are stable and which are unstable. (Same warning as before.) You do not have to find and classify the periodic points; just the fixed points.

The fixed points are exactly as in part (a), only now $f(x) = x + \frac{1}{2} \sin \left( \frac{x^2}{\pi} \right)$, so $f'(x) = 1 + \frac{x}{\pi} \cos \left( \frac{x^2}{\pi} \right) = 1 \pm \sqrt{n}$. This is at least 1 in magnitude unless $\sqrt{n} < 2x$ and we are subtracting. The only three stable fixed points are $\pi$, $-\sqrt{2}\pi$, and $\sqrt{3}\pi$. The fixed points $x = -2\pi$ and $x = 0$ are neutral, and all other fixed points are unstable.