1. Consider the nonlinear system of differential equations

\[
\frac{dx_1}{dt} = x_1(1 - x_1 - 2x_2) \\
\frac{dx_2}{dt} = -x_2(1 - \frac{x_1}{2})
\]

a) Find the fixed points.

Either \( x_1 = 0 \) or \( 1 - x_1 - 2x_2 = 0 \), and either \( x_2 = 0 \) or \( 1 - x_1/2 = 0 \). Since you can’t have \( x_1 \) and \( 1 - x_1/2 \) both equaling zero, there are three possibilities, namely \((0,0), (1,0)\) and \((2, -\frac{1}{2})\).

b) For each fixed point, find a linear system of differential equations that approximates the system near the fixed point.

For each fixed point \( \vec{a} \), let \( \vec{y} = \vec{x} - \vec{a} \), and we have the approximate equations \( \frac{d\vec{a}}{dt} = A\vec{y} \), where \( A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \), evaluated at \( \vec{a} \). This gives

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -2 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} -2 & -4 \\ -\frac{1}{4} & 0 \end{pmatrix}
\]

at the three fixed points, respectively.

c) For each fixed point, indicate whether the point is a source, sink, saddle point, spiral (in or out?), or is borderline.

The eigenvalues of the first matrix are \( \pm 1 \), so \((0,0)\) is a saddle. The eigenvalues of the second are \(-1\) and \(-\frac{1}{2}\), so \((1,0)\) is a sink. The eigenvalues of the third are \(-1 \pm \sqrt{2}\), so \((2, -\frac{1}{2})\) is a saddle.

2. Consider the differential equation \( y'' + \sin(x)y' + \cos(x)y = 0 \). Recall that \( \sin(x) = x + O(x^3) \) and \( \cos(x) = 1 - \frac{x^2}{2} + O(x^4) \). We seek solutions of the form \( y = \sum_{n=0}^{\infty} a_n x^n \).

a) If \( y(0) = 1 \) and \( y'(0) = 0 \), find \( a_0, a_1, a_2, a_3 \) and \( a_4 \).

\[
y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + O(x^5), \ \text{so} \ \ y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + O(x^4) \ \text{and} \ \ y''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + O(x^3). \ \text{Since} \ \sin(x) = x + O(x^3), \ \sin(x)y' = a_1 x + 2a_2 x^2 + O(x^3). \ \text{Since} \ \cos(x) = 1 - \frac{x^2}{2} + O(x^4), \ \cos(x)y = a_0 + a_1 x + (a_2 - \frac{a_0}{2}) x^2 + O(x^3). \ \text{The entire right hand side of the differential equation can be written as} \]

\[
y'' + \sin(x)y' + \cos(x)y = 2a_2 + a_0 + (2a_1 + 6a_3) x + (12a_4 + 3a_2 - \frac{a_0}{2}) x^2 + O(x^3). \ \text{This implies that} \ a_2 = -a_0/2, \ a_3 = -a_1/3 \ \text{and} \ a_4 = (a_0 - 6a_2)/24 = a_0/6.
\]
If $y(0) = 1$ and $y'(0) = 0$, then $a_0 = 1$, $a_1 = 0$, $a_2 = -\frac{1}{2}$, $a_3 = 0$, and $a_4 = \frac{1}{6}$. In other words, $y = 1 - \frac{x^2}{2} + \frac{x^4}{6} + O(x^5)$.

b) If $y(0) = 0$ and $y'(0) = 1$, find $a_0$, $a_1$, $a_2$, $a_3$ and $a_4$.

If $y(0) = 0$ and $y'(0) = 1$, then $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = -1/3$, and $a_4 = 0$. In other words, $y = x - \frac{x^3}{3} + O(x^5)$.

3. Now consider the differential equation $x^2y'' + xy' + (x^2 - 2)y = 0$. (This is a special case of Bessel’s equation.) For $x > 0$, we seek solutions of the form $y = x^r(a_1 + a_1 x + a_2 x^2 + \cdots)$, with $a_0$ nonzero.

a) For what values of $r$ do such solutions exist?

Our equation for $r$ is $r^2 - 2 = 0$, so $r = \pm \sqrt{2}$.

b) For the largest value of $r$, find a recursion relation expressing $a_n$ in terms of $a_0$, $a_1$, $\ldots$, $a_{n-1}$.

If $y = x^r \sum a_n x^n = \sum a_n x^{n+r}$, then $xy' = x^r \sum (n+r) a_n x^n$, $x^2 y'' = x^r \sum (n+r)(n+r-1) a_n x^n$, and $(x^2 - 2)y = x^r \sum (a_{n-2} - 2a_n) x^n$ (with $a_{-2} = a_{-1} = 0$).

Plugging this into the equation gives $a_n [(n + r)^2 - 2] + a_{n-2} = 0$. For $n = 0$ this implies that $r = \pm \sqrt{2}$. For $n = 1$ it implies that $a_1 = 0$, and hence that all the odd $a_n$ vanish. For higher $n$ we have $a_n = \frac{-a_{n-2}}{(n+r)^2 - 2}$.

b) For the largest value of $r$, set $a_0 = 1$ and find $a_1$, $a_2$ and $a_3$.

$$a_1 = a_3 = 0 \text{ and } a_2 = -\frac{1}{(2 + \sqrt{2})^2} = -\frac{1}{2} + \frac{\sqrt{2}}{4}.$$  

4. On the interval $[0, 1]$, we seek to expand the function

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

as a Fourier sine series $f(x) = \sum_{n=1}^{\infty} c_n \sin(n \pi x)$.

a) Find $c_n$ for all $n$. [You may find the identity $\int x \sin(ax)dx = \frac{\sin(ax)}{a} - \frac{x \cos(ax)}{a}$ to be useful]

$$c_n = 2 \int_0^1 x \sin(n \pi x)dx = -\frac{2 \cos(n \pi)}{n \pi}. \text{ This equals } 2/n \pi \text{ if } n \text{ is odd and } -2/n \pi \text{ if } n \text{ is even.}$$

b) Evaluate this series at $x = 1/2$ to obtain a formula for $\pi$ as an infinite sum of rational numbers.

$$\frac{1}{2} = f(\frac{1}{2}) = \sum c_n \sin(n \pi / 2) = c_1 - c_3 + c_5 - c_7 + \cdots = \frac{2}{\pi} \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right),$$

so $\pi = 4 \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right)$. 