M427K Third Midterm Exam, April 30, 2008

1. Consider the nonlinear system of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(1 - x_1 - 2x_2) \\ \frac{dx_2}{dt} &= -x_2(1 - \frac{x_1}{2}) \end{aligned}$$

a) Find the fixed points.

b) For each fixed point, find a linear system of differential equations that approximates the system near the fixed point.

c) For each fixed point, indicate whether the point is a source, sink, saddle point, spiral (in or out?), or is borderline.

2. Consider the differential equation $y'' + \sin(x)y' + \cos(x)y = 0$. Recall that $\sin(x) = x + O(x^3)$ and $\cos(x) = 1 - \frac{x^2}{2} + O(x^4)$. We seek solutions of the form $y = \sum_{n=0}^{\infty} a_n x^n$.

a) If $y(0) = 1$ and $y'(0) = 0$, find a_0, a_1, a_2, a_3 and a_4.

b) If $y(0) = 0$ and $y'(0) = 1$, find a_0, a_1,a_2, a_3 and a_4.

3. Now consider the differential equation $x^2y'' + xy' + (x^2 - 2)y = 0$. (This is a special case of Bessel's equation.) For $x > 0$, we seek solutions of the form $y = x^r(a_1 + a_1x + a_2x^2 + \cdots)$, with a_0 nonzero.

a) For what values of r do such solutions exist?

b) For the largest value of r, find a recursion relation expressing a_n in terms of a_0, $a_1, \ldots,$ a_{n-1}.

b) For the largest value of r, set $a_0 = 1$ and find a_1, a_2 and a_3.

4. On the interval $[0, 1]$, we seek to expand the function

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

as a Fourier sine series $f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$.

a) Find c_n for all n. [You may find the identity $\int x \sin(ax)dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a}$ to be useful]

b) Evaluate this series at $x = 1/2$ to obtain a formula for π as an infinite sum of rational numbers.