1. Consider the nonlinear system of differential equations

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1(1 - x_1 - 2x_2) \\
\frac{dx_2}{dt} &= -x_2(1 - \frac{x_1}{2})
\end{align*}
\]

a) Find the fixed points.
b) For each fixed point, find a linear system of differential equations that approximates the system near the fixed point.
c) For each fixed point, indicate whether the point is a source, sink, saddle point, spiral (in or out?), or is borderline.

2. Consider the differential equation

\[y'' + \sin(x)y' + \cos(x)y = 0.\]

Recall that \(\sin(x) = x + O(x^3)\) and \(\cos(x) = 1 - \frac{x^2}{2} + O(x^4)\). We seek solutions of the form \(y = \sum_{n=0}^{\infty} a_n x^n\).

a) If \(y(0) = 1\) and \(y'(0) = 0\), find \(a_0\), \(a_1\), \(a_2\), \(a_3\) and \(a_4\).
b) If \(y(0) = 0\) and \(y'(0) = 1\), find \(a_0\), \(a_1\), \(a_2\), \(a_3\) and \(a_4\).

3. Now consider the differential equation \(x^2y'' + xy' + (x^2 - 2)y = 0\). (This is a special case of Bessel’s equation.) For \(x > 0\), we seek solutions of the form \(y = x^r(a_1 + a_1 x + a_2 x^2 + \cdots)\), with \(a_0\) nonzero.

a) For what values of \(r\) do such solutions exist?
b) For the largest value of \(r\), find a recursion relation expressing \(a_n\) in terms of \(a_0, a_1, \ldots, a_{n-1}\).
b) For the largest value of \(r\), set \(a_0 = 1\) and find \(a_1, a_2\) and \(a_3\).

4. On the interval \([0, 1]\), we seek to expand the function

\[f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x = 1 \end{cases}\]

as a Fourier sine series \(f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)\).

a) Find \(c_n\) for all \(n\). [You may find the identity \(\int x \sin(ax)dx = \frac{\sin(ax)}{a^2} - \frac{x \cos(ax)}{a}\) to be useful]
b) Evaluate this series at \(x = 1/2\) to obtain a formula for \(\pi\) as an infinite sum of rational numbers.